

SINGULAR MASAS AND MEASURE-MULTIPLICITY INVARIANT

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ABSTRACT. In this paper we study relations between the *left-right-measure* and properties of singular masas. Part of the analysis is mainly concerned with masas for which the *left-right-measure* is the class of product measure. We provide examples of Tauer masas in the hyperfinite II_1 factor whose *left-right-measure* is the class of Lebesgue measure. We show that for each subset $S \subseteq \mathbb{N}$, there exist uncountably many pairwise non conjugate singular masas in the free group factors with *Pukánszky invariant* $S \cup \{\infty\}$.

1. INTRODUCTION AND PRELIMINARIES

Throughout the entire paper, \mathcal{M} will denote a separable II_1 factor equipped with its faithful normal tracial state τ . This trace gives rise to a Hilbert norm on \mathcal{M} , given by $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$, $x \in \mathcal{M}$. The Hilbert space completion of \mathcal{M} with respect to $\|\cdot\|_2$ is denoted by $L^2(\mathcal{M})$. Let \mathcal{M} act on $L^2(\mathcal{M})$ via left multiplication. Let $A \subset \mathcal{M}$ be a maximal abelian self-adjoint subalgebra (masa). Dixmier in [5] defined the group of *normalizing unitaries* (or *normalizer*) of A to be the set

$$N(A) = \{u \in \mathcal{U}(\mathcal{M}) : uAu^* = A\},$$

where $\mathcal{U}(\mathcal{M})$ denotes the unitary group of \mathcal{M} . He called

- (i) A to be *regular* (also *Cartan*) if $N(A)'' = \mathcal{M}$,
- (ii) A to be *semiregular* if $N(A)''$ is a subfactor of \mathcal{M} ,
- (iii) A to be *singular* if $N(A) \subset A$.

Two masas A, B of \mathcal{M} are said to be *conjugate*, if there is an automorphism θ of \mathcal{M} such that $\theta(A) = B$. If there is a unitary $u \in \mathcal{M}$ such that $uAu^* = B$, then A and B are called *unitarily (inner) conjugate*. One of the most fundamental problem regarding masas is to decide the conjugacy of two masas. The most successful invariant so far in this regard is the *Pukánszky invariant* [22]. Nevertheless, it is not a complete invariant.

The *measure-multiplicity invariant* of masas in II_1 factors was studied in [8, 12, 14]. It was used in [8] to distinguish two masas with the same *Pukánszky invariant*. It is a stronger invariant than the *Pukánszky invariant*. It has two main components, a measure class (*left-right-measure*) and a multiplicity function, which together encode the structure of the standard Hilbert space as an associated bimodule. In this paper, we study analytical relations between the *left-right-measure* and properties of singular masas. We focus on the following question: To what extent does the standard Hilbert space as a natural bimodule remember properties of the masa. In [12], we established that *left-right-measure* has all information to measure the size of $N(A)$ (see Thm. 5.5 [12]).

In this paper, we consider different kinds of singular masas. We introduce a condition on masas which forces vigorous mixing properties. Such masas are automatically strongly mixing [9] (for a proof see Thm. 9.2 [1]) and consequently singular. We show that if A is such a masa in \mathcal{M} with singleton multiplicity, then the Hilbert space $L^2(\mathcal{M}) \ominus L^2(A)$ as a natural A, A -bimodule is a direct sum of copies of $L^2(A) \otimes L^2(A)$, i.e., we show that its *left-right-measure* is the class of product measure. We also present a converse to the foresaid statement. The arguments required to prove this statement show that, if $B \subset A$ is diffuse, then $L^2(\mathcal{M}) \ominus L^2(B' \cap \mathcal{M})$ as a B, B -bimodule is a direct sum of copies of submodules of $L^2(B) \otimes L^2(B)$. There is an abundance of such masas in the hyperfinite II_1 factor, but there are fewer examples of such masas, if in addition we demand that such a masa has a bicyclic vector. We also study the *left-right-measure* of Γ and non- Γ singular masas. In particular, we show that under certain extra assumption on central sequences, the presence of central sequences in a masa can be related to *rigid measures*. Examples of such masas come from Ergodic theory.

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The following question asked by Banach is a long standing open problem in Ergodic theory. Does there exist a simple measure preserving (m.p.) automorphism with pure Lebesgue spectrum? It is implicit in the above question that one is asking about an action of \mathbb{Z} . Translated to operator algebras this means (see for instance [14]), whether there is a way to construct the hyperfinite II_1 factor as $L^\infty(X, \mu) \rtimes \mathbb{Z}$, where $L(\mathbb{Z})$ is a *simple masa* whose *left-right-measure* is the class of product measure. The term ‘simple’ of course means simple multiplicity or equivalently, the existence of a bicyclic vector of A .

We provide an example of such a Tauer masa in the hyperfinite II_1 factor. All Tauer masas are simple [32]. We do not know if this example arises from an action of integers or any other group action. But quite surprisingly Banach’s problem has an easy and affirmative answer if we change the group. Using the methods developed in [8], we show that for each subset $S \subseteq \mathbb{N}$ (could be empty), there are uncountably many pairwise non conjugate singular masas in the free group factors with *Pukánszky invariant* $S \cup \{\infty\}$.

This paper is organized as follows. We provide the background material in this section itself. In §2, we study masas for which the *left-right-measure* is the class of product measure. §3 is devoted to Tauer masas. §4 contains partial results regarding the *left-right-measure* of Γ and non- Γ masas. In §5, we exhibit examples of singular masas in free group factors.

Let J denote the Tomita’s modular operator on $L^2(\mathcal{M})$, obtained by extending the densely defined map $J : \mathcal{M} \mapsto \mathcal{M}$ by $Jx = x^*$. The image of a L^2 vector ζ under J will be denoted by ζ^* . Let $e_A : L^2(\mathcal{M}) \mapsto L^2(A)$ be the Jones projection associated to A . Denote $\mathcal{A} = (A \cup JAJ)''$. It is known that $e_A \in \mathcal{A}$ (Thm. 3.1 [21]). Let \mathbb{E}_A denote the unique, normal, trace preserving conditional expectation from \mathcal{M} on to A . The conditional expectation \mathbb{E}_A and the trace extends to $L^1(\mathcal{M})$ in a continuous fashion (see §B.5 [27]). With abuse of notation, we will write $e_A(\zeta) = \mathbb{E}_A(\zeta)$ for L^1 and L^2 vectors. Similarly, we will use the same symbol τ to denote its extension. This will be clear from the context and will cause no confusion. This work relies on direct integrals. For standard results on direct integrals we refer the reader to [6]. Throughout the entire paper \mathbb{N}_∞ will denote the set $\mathbb{N} \cup \{\infty\}$. For a set X , we will write $\Delta(X)$ to denote the diagonal of $X \times X$.

Definition 1.1. Given a type I von Neumann algebra B , we shall write $\text{Type}(B)$ for the set of all those $n \in \mathbb{N}_\infty$ such that B has a nonzero component of type I_n .

Definition 1.2. [22] The *Pukánszky invariant* of $A \subset \mathcal{M}$, denoted by $\text{Puk}(A)$ (or $\text{Puk}_{\mathcal{M}}(A)$ when the containing factor is ambiguous) is $\text{Type}(\mathcal{A}'(1 - e_A))$.

Definition 1.3. [8, 12, 14] The *measure-multiplicity invariant* of $A \subset \mathcal{M}$, denoted by $m.m.(A)$, is the equivalence class of quadruples $(X, \lambda_X, [\eta|_{\Delta(X)^c}], m|_{\Delta(X)^c})$ under the equivalence relation $\sim_{m.m.}$, where,

(i) X is a compact Hausdorff space such that $C(X)$ is an unital, norm separable, *w.o.t* dense subalgebra of A ,

(ii) λ_X is the Borel probability measure obtained by restricting the trace τ on $C(X)$,

and,

(iii) $\eta|_{\Delta(X)^c}$ is the measure on $X \times X$ concentrated on $\Delta(X)^c$, and

(iv) $m|_{\Delta(X)^c}$ is the multiplicity function restricted to $\Delta(X)^c$,

obtained from the direct integral decomposition of $L^2(\mathcal{M}) \ominus L^2(A)$, so that $\mathcal{A}(1 - e_A)$ is the algebra of diagonalizable operators with respect to this decomposition, the equivalence $\sim_{m.m.}$ being, $(X, \lambda_X, [\eta|_{\Delta(X)^c}], m|_{\Delta(X)^c}) \sim_{m.m.} (Y, \lambda_Y, [\eta|_{\Delta(Y)^c}], m|_{\Delta(Y)^c})$, if and only if, there exists a Borel isomorphism $F : X \mapsto Y$ such that,

$$\begin{aligned} F_*\lambda_X &= \lambda_Y, \\ (F \times F)_*[\eta|_{\Delta(X)^c}] &= [\eta|_{\Delta(Y)^c}], \\ m|_{\Delta(X)^c} \circ (F \times F)^{-1} &= m|_{\Delta(Y)^c}, \eta|_{\Delta(Y)^c} \text{ a.e.} \end{aligned}$$

It is easy to see that the *Pukánszky invariant* of $A \subset \mathcal{M}$ is the set of essential values of the multiplicity function in Defn. 1.3. The measure class $[\eta|_{\Delta(X)^c}]$ in Defn. 1.3 is said to be the *left-right-measure* of A . Both $m.m.(A)$ and $\text{Puk}(A)$ are invariants of the masa under automorphisms of the factor \mathcal{M} . Given a pair of masas, the *mixed Pukánszky invariant* was introduced in [33]. Analogous to the *mixed Pukánszky invariant*, one can define the *joint-measure-multiplicity invariant* for pair of masas A and B , by considering the direct integral decomposition of $L^2(\mathcal{M})$ with respect to $(A \cup JBJ)''$.

For details check Ch. V [11]. Such invariants play a role in questions concerning unitary conjugacy of masas.

In some cases, it is necessary to have a direct integral decomposition of $L^2(\mathcal{M})$. These are situations when one considers tensors of masas. In these cases, the information on the diagonal $\Delta(X)$ is to be supplied. Following §2.3 of [12] note that

$$L^2(\mathcal{M}) \cong \int_{X \times X}^{\oplus} \mathcal{H}_{t,s} d(\eta|_{\Delta(X)^c} + \tilde{\Delta}_* \lambda_X)(t, s),$$

where $\tilde{\Delta} : X \mapsto X \times X$ by $t \mapsto (t, t)$, $\mathcal{H}_{t,t} = \mathbb{C}$ for λ_X almost all t , $\mathcal{H}_{t,s}$ depends on the *Pukánszky invariant* and \mathcal{A} is diagonalizable with respect to this decomposition. In such cases, we will also call $[\eta|_{\Delta(X)^c} + \tilde{\Delta}_* \lambda_X]$ to be the *left-right-measure* of A . It is to be understood that, when we consider direct integrals or make statements about diagonalizability, we need to complete the measures under consideration.

For a masa $A \subset \mathcal{M}$, fix a compact Hausdorff space X such that $C(X) \subset A$ is an unital, norm separable and *w.o.t* dense C^* subalgebra. Let λ denote the tracial measure on X . For $\zeta_1, \zeta_2 \in L^2(\mathcal{M})$, let $\kappa_{\zeta_1, \zeta_2} : C(X) \otimes C(X) \mapsto \mathbb{C}$ be the linear functional defined by,

$$\kappa_{\zeta_1, \zeta_2}(a \otimes b) = \langle a \zeta_1 b, \zeta_2 \rangle, \quad a, b \in C(X).$$

Then $\kappa_{\zeta_1, \zeta_2}$ induces an unique complex Radon measure η_{ζ_1, ζ_2} on $X \times X$ given by,

$$(1.1) \quad \kappa_{\zeta_1, \zeta_2}(a \otimes b) = \int_{X \times X} a(t)b(s) d\eta_{\zeta_1, \zeta_2}(t, s),$$

and $\|\eta_{\zeta_1, \zeta_2}\|_{t.v} = \|\kappa_{\zeta_1, \zeta_2}\|$, where $\|\cdot\|_{t.v}$ denotes the total variation norm of measures.

We will write $\eta_{\zeta, \zeta} = \eta_{\zeta}$. Note that η_{ζ} is a positive measure for all $\zeta \in L^2(\mathcal{M})$. It is easy to see that the following polarization type identity holds:

$$(1.2) \quad 4\eta_{\zeta_1, \zeta_2} = (\eta_{\zeta_1 + \zeta_2} - \eta_{\zeta_1 - \zeta_2}) + i(\eta_{\zeta_1 + i\zeta_2} - \eta_{\zeta_1 - i\zeta_2}).$$

Note that the decomposition of η_{ζ_1, ζ_2} in Eq. (1.2) need not be its Hahn decomposition in general, but

$$(1.3) \quad 4|\eta_{\zeta_1, \zeta_2}| \leq (\eta_{\zeta_1 + \zeta_2} + \eta_{\zeta_1 - \zeta_2}) + (\eta_{\zeta_1 + i\zeta_2} + \eta_{\zeta_1 - i\zeta_2}) = 4(\eta_{\zeta_1} + \eta_{\zeta_2}).$$

So

$$(1.4) \quad |\eta_{\zeta_1, \zeta_2}| \leq \eta_{\zeta_1} + \eta_{\zeta_2}.$$

To understand the relation between properties of masas and their *left-right-measure*, disintegration of measures will be used, for which we refer the reader to §2 of [2]. Let T be a measurable map from (X, σ_X) to (Y, σ_Y) , where σ_X, σ_Y are σ -algebras of subsets of X, Y respectively. Let β be a σ -finite measure on σ_X and μ a σ -finite measure on σ_Y . Here β is the measure to be disintegrated and μ is often the push forward measure $T_*\beta$, although other possibilities for μ is allowed.

Definition 1.4. [2] We say that β has a disintegration $\{\beta^t\}_{t \in Y}$ with respect to T and μ or a (T, μ) -disintegration if:

(i) β^t is a σ -finite measure on σ_X concentrated on $\{T = t\}$ (or $T^{-1}\{t\}$), i.e., $\beta^t(\{T \neq t\}) = 0$, for μ -almost all t ,

and, for each nonnegative σ_X -measurable function f on X :

(ii) $t \mapsto \beta^t(f)$ is σ_Y -measurable.

(iii) $\beta(f) = \mu^t(\beta^t(f)) \stackrel{\text{defn}}{=} \int_Y \beta^t(f) d\mu(t)$.

If β in Defn. 1.4 is a complex measure, then the disintegration of β is obtained by decomposing it into a linear combination of four positive measures, using the Hahn decomposition of its real and imaginary parts.

Notation: The disintegrated measures are usually written with a subscript $t \mapsto \beta_t$ in the literature. But in this paper, we will use the superscript notation $t \mapsto \beta^t$ to denote them. The (π_1, λ) -disintegration of measures on $X \times X$ will be indexed by the variable t and the (π_2, λ) -disintegration will be indexed by the variable s , where $\pi_i : X \times X \mapsto X$, $i = 1, 2$, are the coordinate projections. We will only consider the (π_i, λ) -disintegrations of the measures $\eta_{\zeta}, \eta_{\zeta_1, \zeta_2}$ defined in Eq. (1.1). These disintegrations exist from Thm. 3.2 [12] (also see Thm. 1 [2]). The measure η_{ζ}^t is concentrated on $\{t\} \times X$ and the measure η_{ζ}^s is concentrated on $X \times \{s\}$ for λ almost all t, s respectively. We will denote by $\tilde{\eta}_{\zeta}^t$ the restriction of the measure η_{ζ}^t on $\{t\} \times X$. Similarly define $\tilde{\eta}_{\zeta}^s$. Thus, $\tilde{\eta}_{\zeta}^t, \tilde{\eta}_{\zeta}^s$ can be

regarded as measures on X .

The *left-right-measure* $[\eta]$ of A has the following property. If $\theta : X \times X \mapsto X \times X$ is the flip map i.e., $\theta(t, s) = (s, t)$, then $\theta_*\eta \ll \eta \ll \theta_*\eta$ (see Lemma 2.9 [12]). In fact, it is possible to obtain a choice of η for which $\theta_*\eta = \eta$. So in most of the analysis, we will only state or prove results with respect to the (π_1, λ) -disintegration. An analogous statement with respect to the (π_2, λ) -disintegration is also possible. We will only work with finite or probability measures.

2. THE PRODUCT CLASS

It is not always easy to describe the properties of a singular masa based on its *left-right-measure*. However, we can write interesting properties of masas when the *left-right-measure* is the class of product measure. Such masas are singular Thm. 5.5 [12]. Examples of such masas are easy to obtain in many situations and many known masas, for example, the single generator masas in the free group factors, the masas that arise out of Bernoulli shift actions of countable discrete abelian groups belong to this class. In this section, we shall give analytical conditions for the *left-right-measure* of a masa to be the class of product measure.

Let λ denote the Lebesgue measure on $[0, 1]$ so that $A \cong L^\infty([0, 1], \lambda)$. Then λ is the tracial measure. Let $[\eta]$ denote the *left-right-measure* of A . We assume that η is a probability measure on $[0, 1] \times [0, 1]$ and $\eta(\Delta([0, 1])) = 0$.

‘The *left-right-measure* of any masa in the free group factors contains a part of $\lambda \otimes \lambda$ as a summand’. This statement of Voiculescu is one of the most important theorem in the subject (Cor. 7.6 [30] applied to a system of free semicirculars does the job.) This is the precise reason for the absence of Cartan subalgebras in the free group factors. In many cases, the *left-right-measures* are difficult to calculate. So we need conditions in terms of operators that characterize the Lebesgue class. The following is the main result of this section. It will be proved later in this section.

Theorem 2.1. *The left-right-measure of a masa $A \subset \mathcal{M}$ is the class of product measure, if there exists a set $S \subset \mathcal{M}$ such that $\mathbb{E}_A(x) = 0$ for all $x \in S$ and*

- (i) *the linear span of S is dense in $L^2(\mathcal{M}) \ominus L^2(A)$,*
- (ii) *there is an orthonormal basis $\{v_n\}_{n=1}^\infty \subset A$ of $L^2(A)$ such that*

$$\sum_{n=1}^{\infty} \|\mathbb{E}_A(xv_nv_n^*)\|_2^2 < \infty \text{ for all } x \in S,$$

- (iii) *there is a nonzero vector $\zeta \in L^2(\mathcal{M}) \ominus L^2(A)$ such that $\mathbb{E}_A(\zeta u^n \zeta^*) = 0$ for all $n \neq 0$, where u is a Haar unitary generator of A .*

We do not know whether the conditions in Thm. 2.1 are necessary for the same conclusion to hold. In Thm. 2.7, we provide an analogous condition which is necessary for the *left-right-measure* to be of the product class. In general, it is of interest to know whether there exist masas for which $\eta \ll \lambda \otimes \lambda$ but $[\eta] \neq [\lambda \otimes \lambda]$. Note that the sum in Thm. 2.1 is independent of the choice of the orthonormal basis. This just follows by expanding elements of one orthonormal basis with respect to another. Hence by making similar arguments, (iii) in Thm. 2.1 holds for any Haar unitary generator of A , if it holds for one Haar unitary generator. Conditions (i) and (ii) in Thm. 2.1 forces that $\eta \ll \lambda \otimes \lambda$. To assure $\lambda \otimes \lambda \ll \eta$ we need condition (iii) (Cor. 2.8). As it will become clear, these conditions are analogous to knowing a measure from the information of its Fourier coefficients. Condition (iii) is an analogue of the fact that the Fourier coefficients of λ are 0 except for the zeroth coefficient. In this sense the operators $\mathbb{E}_A(xv_nv_n^*)$ can be thought of as the ‘Fourier coefficients of the bimodule $\overline{Ax A}^{\|\cdot\|_2}$ ’.

In order to motivate the conditions in Thm. 2.1, we cite some examples. Conditions (i) and (ii) first appeared in the study of radial masas in the free group factors (see Thm. 3.1 [25]). In this case, the natural choice of the orthonormal basis is $v_n = \frac{\chi_n}{\|\chi_n\|_2}$, $n \geq 0$, where $\chi_n = \sum_{w:|w|=n} w$. The set S consists of $w - \mathbb{E}_A(w)$, $1 \neq w \in \mathbb{F}_k$, $k \geq 2$. Single generator masas in the free group factors clearly satisfies all the three conditions. In fact, for all masas exhibited in §6 there is a nonzero vector for which (iii) in Thm. 2.1 is satisfied. However, in those examples (i) won’t be satisfied.

The second class of examples comes from inclusion of groups. Suppose Γ is a countable discrete icc group with $\Lambda < \Gamma$, such that $L(\Lambda) \subset L(\Gamma)$ is a singular masa. Assume further that Λ is malnormal in Γ . It is obvious that (i), (ii) and (iii) hold (Thm. 9.5, Cor. 9.8 [1]). Similar class of examples consists of freely complemented masas.

Next class of examples comes from ergodic theory. Consider a countable infinite abelian group Γ acting on $\prod_{\Gamma}(X, \mu)$ by Bernoulli shift, where (X, μ) is a probability space with at least two points. Then $L(\Gamma) \subset L^\infty(\prod_{\Gamma}(X, \mu)) \rtimes \Gamma$ is a singular masa whose *left-right-measure* is the class of product Haar measure on $\widehat{\Gamma} \times \widehat{\Gamma}$ (Prop. 3.1 [14]). For any function f on the γ -th copy, $\gamma \in \Gamma$, such that $\mu(f) = 0$ one has

$$\mathbb{E}_{L(\Gamma)}(fu_{\gamma'}f^*) = \tau(f\alpha_{\gamma'}(f^*))u_{\gamma'}, \gamma' \in \Gamma,$$

where $u_{\gamma'}$ are the canonical group unitaries in the crossed product, τ is the tracial state and $\alpha_{\gamma'}$ denotes the automorphism corresponding to γ' . It follows that $\sum_{\gamma' \in \Gamma} \|\mathbb{E}_{L(\Gamma)}(fu_{\gamma'}f^*)\|_2^2$ is finite. It is now clear that (i) and (ii) of Thm. 2.1 hold. Since the maximal spectral type of the Bernoulli action is the normalized Haar measure on $\widehat{\Gamma}$, any function $f \in L^2(\prod_{\Gamma}(X, \mu))$ for which the maximal spectral type is attained, satisfies $\mathbb{E}_{L(\Gamma)}(fu_{\gamma}f^*) = 0$ for all $\gamma \neq 1$. The last statement is equivalent to (iii).

In order to prove Thm. 2.1, we need to prove some auxiliary lemmas.

Lemma 2.2. *Let $\zeta_1, \zeta_2 \in L^2(\mathcal{M})$ be such that $\mathbb{E}_A(\zeta_1) = 0 = \mathbb{E}_A(\zeta_2)$. Let η_{ζ_1, ζ_2} denote the Borel measure on $[0, 1] \times [0, 1]$ defined in Eq. (1.1).*

1°. *Then η_{ζ_1, ζ_2} admits (π_i, λ) -disintegrations $[0, 1] \ni t \mapsto \eta_{\zeta_1, \zeta_2}^t$ and $[0, 1] \ni s \mapsto \eta_{\zeta_1, \zeta_2}^s$, where π_i , $i = 1, 2$, denotes the coordinate projections. Moreover,*

$$\eta_{\zeta_1, \zeta_2}^t([0, 1] \times [0, 1]) = \mathbb{E}_A(\zeta_1 \zeta_2^*)(t), \quad \lambda \text{ a.e.}$$

2°. *Let $f \in C[0, 1]$. Then the functions $[0, 1] \ni t \mapsto \eta_{\zeta_1, \zeta_2}^t(1 \otimes f)$, $[0, 1] \ni s \mapsto \eta_{\zeta_1, \zeta_2}^s(f \otimes 1)$ are in $L^1([0, 1], \lambda)$.*

If $\zeta_i \in \mathcal{M}$ for $i = 1, 2$, then $[0, 1] \ni t \mapsto \eta_{\zeta_1, \zeta_2}^t(1 \otimes f)$, $[0, 1] \ni s \mapsto \eta_{\zeta_1, \zeta_2}^s(f \otimes 1)$ are in $L^\infty([0, 1], \lambda)$.

3°. *Let $b, w \in C[0, 1]$. If $\mathbb{E}_A(\zeta_1 w \zeta_2^*) \in L^2(A)$, then*

$$\|\mathbb{E}_A(b \zeta_1 w \zeta_2^*)\|_2^2 = \int_0^1 |b(t)|^2 |\eta_{\zeta_1, \zeta_2}^t(1 \otimes w)|^2 d\lambda(t).$$

Proof. 1°. That η_{ζ_1, ζ_2} admits the stated disintegrations follows from Eq. (1.2), Lemma 5.7 [8] and Lemma 2.9 [12]. The next statement in 1° follows from an argument similar to the proof of Lemma 6.1 [12].

2°. From Eq. (1.3), $|\eta_{\zeta_1, \zeta_2}|$ admits (π_i, λ) -disintegrations. Use Hahn decomposition of measures and Lemma 3.6 [12] to see that $|\eta_{\zeta_1, \zeta_2}|^t = |\eta_{\zeta_1, \zeta_2}^t|$ for λ almost all t . The function $t \mapsto \eta_{\zeta_1, \zeta_2}^t(1 \otimes f)$ is clearly measurable from Defn. 1.4, and from Eq. (1.4) we have

$$\begin{aligned} \int_0^1 |\eta_{\zeta_1, \zeta_2}^t(1 \otimes f)| d\lambda(t) &\leq \|f\| \int_0^1 |\eta_{\zeta_1, \zeta_2}^t|([0, 1] \times [0, 1]) d\lambda(t) \\ &\leq \|f\| \left(\int_0^1 \eta_{\zeta_1}^t([0, 1] \times [0, 1]) d\lambda(t) + \int_0^1 \eta_{\zeta_2}^t([0, 1] \times [0, 1]) d\lambda(t) \right) \\ &= \|f\| (\|\mathbb{E}_A(\zeta_1 \zeta_1^*)\|_1 + \|\mathbb{E}_A(\zeta_2 \zeta_2^*)\|_1) < \infty. \end{aligned}$$

When $\zeta_i \in \mathcal{M}$ a similar argument shows that the stated functions are in $L^\infty([0, 1], \lambda)$.

3°. Since

$$\begin{aligned} \infty &> \sup_{a \in C[0, 1], \|a\|_2 \leq 1} \left| \int_0^1 a(t) b(t) \mathbb{E}_A(\zeta_1 w \zeta_2^*)(t) d\lambda(t) \right| &= \sup_{a \in C[0, 1], \|a\|_2 \leq 1} |\tau(ab \mathbb{E}_A(\zeta_1 w \zeta_2^*))| \\ &= \sup_{a \in C[0, 1], \|a\|_2 \leq 1} |\tau(ab \zeta_1 w \zeta_2^*)| \\ &= \sup_{\substack{a \in C[0, 1] \\ \|a\|_2 \leq 1}} \left| \int_0^1 a(t) b(t) \eta_{\zeta_1, \zeta_2}^t(1 \otimes w) d\lambda(t) \right| \end{aligned}$$

and $t \xrightarrow{g} b(t)\eta_{\zeta_1, \zeta_2}^t(1 \otimes w)$ is in $L^1(\lambda)$, so g is in $L^2(\lambda)$ and

$$\|\mathbb{E}_A(b\zeta_1 w \zeta_2^*)\|_2^2 = \int_0^1 |b(t)|^2 |\eta_{\zeta_1, \zeta_2}^t(1 \otimes w)|^2 d\lambda(t).$$

□

Let $w := \{w_n\}_{n=1}^\infty \subset C[0, 1]$ be an orthonormal basis of $L^2(A)$.

Proposition 2.3. *Let $x_i \in \mathcal{M}$ for $i = 1, 2$, be such that $\mathbb{E}_A(x_i) = 0$. Let us suppose that*

$$\sum_{n=1}^\infty \|\mathbb{E}_A(x_1 w_n x_2^*)\|_2^2 < \infty.$$

If $w' := \{w'_n\}_{n=1}^\infty$ be an orthonormal sequence in $L^2(A)$ with $w'_n \in C[0, 1]$ for all n , then there is a set $F(w, w') \subset [0, 1]$ which depends on w, w' such that $\lambda(F(w, w')) = 0$ and for all $t \in F(w, w')^c$,

$$\sum_{n=1}^\infty |\eta_{x_1, x_2}^t(1 \otimes w'_n)|^2 \leq \sum_{n=1}^\infty |\eta_{x_1, x_2}^t(1 \otimes w_n)|^2 < \infty.$$

Proof. Note that the hypothesis implies that for any $a \in C[0, 1]$,

$$\sum_{n=1}^\infty \|\mathbb{E}_A(ax_1 w_n x_2^*)\|_2^2 < \infty$$

and this sum is independent of the choice of the orthonormal basis. Therefore, for all $a \in C[0, 1]$,

$$\sum_{n=1}^\infty \|\mathbb{E}_A(ax_1 w'_n x_2^*)\|_2^2 \leq \sum_{n=1}^\infty \|\mathbb{E}_A(ax_1 w_n x_2^*)\|_2^2.$$

Let $r \in A$ be a nonzero projection. Identify r with a measurable subset E_r of $[0, 1]$. We can assume E_r is a Borel set. We claim that

$$(2.1) \quad \int_{E_r} \sum_{n=1}^\infty |\eta_{x_1, x_2}^t(1 \otimes w'_n)|^2 d\lambda(t) \leq \int_{E_r} \sum_{n=1}^\infty |\eta_{x_1, x_2}^t(1 \otimes w_n)|^2 d\lambda(t).$$

If the claim is true, then by standard measure theory arguments we are done.

First assume E_r is a compact set. Choose a sequence of continuous functions f_l such that $0 \leq f_l \leq 1$ and $f_l \downarrow \chi_{E_r}$ pointwise as $l \rightarrow \infty$. Therefore by Lemma 2.2 and monotone convergence theorem, for all l we have,

$$\begin{aligned} \int_0^1 f_l^2(t) \sum_{n=1}^\infty |\eta_{x_1, x_2}^t(1 \otimes w'_n)|^2 d\lambda(t) &= \sum_{n=1}^\infty \|\mathbb{E}_A(f_l x_1 w'_n x_2^*)\|_2^2 \\ &\leq \sum_{n=1}^\infty \|\mathbb{E}_A(f_l x_1 w_n x_2^*)\|_2^2 = \int_0^1 f_l^2(t) \sum_{n=1}^\infty |\eta_{x_1, x_2}^t(1 \otimes w_n)|^2 d\lambda(t). \end{aligned}$$

Passing to limits, we see that Eq. (2.1) is true whenever E_r is compact. Now use regularity of λ to see that Eq. (2.1) is true for all Borel sets of positive measure. □

Let $X = \prod_{n=1}^\infty C[0, 1]$. Equip X with the product topology. Then X is known to be separable and metrizable. Every $\underline{f} \in X$ is a infinite tuple $\underline{f} = (f_1, f_2, \dots)$. Also for a sequence $\underline{f}^{(n)} \in X$, $\underline{f}^{(n)} \rightarrow \underline{f}$ as $n \rightarrow \infty$ implies that $f_k^{(n)} \rightarrow f_k$ in $\|\cdot\|_\infty$ for all $k \in \mathbb{N}$.

Let $\mathcal{O} = \{\underline{f} \in X : \{f_k\}_{k=1}^\infty \text{ is an orthonormal sequence in } L^2([0, 1], \lambda)\}$. Then $\mathcal{O} \subset X$ is a closed set. Note that \mathcal{O} is separable in the product topology.

Proposition 2.4. *Let $x \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0$. Let us suppose that*

$$\sum_{k=1}^\infty \|\mathbb{E}_A(x w_k x^*)\|_2^2 < \infty.$$

Then $\eta_x \ll \lambda \otimes \lambda$.

Proof. Let $\{\underline{w}^{(m)}\}_{m=1}^\infty \subset \mathcal{O}$ be any countable dense set. From Prop. 2.3 and Lemma 2.2, it follows that there is a set $F \subset [0, 1]$ with $\lambda(F) = 0$ such that for $t \in F^c$, η_x^t is a finite measure and

$$\sum_{k=1}^\infty \left| \eta_x^t(1 \otimes w_k^{(m)}) \right|^2 \leq \sum_{k=1}^\infty \left| \eta_x^t(1 \otimes w_k) \right|^2 < \infty$$

for all $m \in \mathbb{N}$. Let $\underline{v} = \{v_k\}_{k=1}^\infty \in \mathcal{O}$. There exists a subsequence $\{\underline{w}^{(m_j)}\}_{j=1}^\infty$ such that $\underline{w}^{(m_j)} \rightarrow \underline{v}$ as $j \rightarrow \infty$. Therefore for $t \in F^c$,

$$\begin{aligned} \sum_{k=1}^\infty \left| \eta_x^t(1 \otimes v_k) \right|^2 &= \sum_{k=1}^\infty \lim_j \left| \eta_x^t(1 \otimes w_k^{(m_j)}) \right|^2 \quad (\text{by Dominated convergence}) \\ &= \sum_{k=1}^\infty \liminf_j \left| \eta_x^t(1 \otimes w_k^{(m_j)}) \right|^2 \\ &\leq \liminf_j \sum_{k=1}^\infty \left| \eta_x^t(1 \otimes w_k^{(m_j)}) \right|^2 \quad (\text{by Fatou's Lemma}) \\ &\leq \sum_{k=1}^\infty \left| \eta_x^t(1 \otimes w_k) \right|^2 < \infty \quad (\text{as } t \in F^c). \end{aligned}$$

Therefore for each $t \in F^c$,

$$(2.2) \quad \sup_{f \in \mathcal{O}} \sum_{k=1}^\infty \left| \eta_x^t(1 \otimes f_k) \right|^2 \leq \sum_{k=1}^\infty \left| \eta_x^t(1 \otimes w_k) \right|^2 < \infty.$$

Fix $t \in F^c$. If $\tilde{\eta}_x^t$ contains a nonzero part which is singular with respect to λ , then the supremum on the left hand side of Eq. (2.2) is infinite. Indeed, for simplicity assume $\tilde{\eta}_x^t \perp \lambda$. Choose a compact set $K \subset [0, 1]$ of almost full $\tilde{\eta}_x^t$ measure such that $\lambda(K) = 0$. Fix a large positive number N . By regularity of λ , there is an open set U containing K such that $\lambda(U) < \frac{1}{N^3}$. Using compactness of K , we can find a finite number of open intervals (a_i, b_i) and small positive numbers δ_i for $i = 1, 2, \dots, m$, such that the open intervals $\{(a_i - \delta_i, b_i + \delta_i)\}_{i=1}^m$ are disjoint and $K \subset \cup_{i=1}^m (a_i, b_i) \subset \cup_{i=1}^m (a_i - \delta_i, b_i + \delta_i) \subset U$. Define

$$f_i(s) = \begin{cases} N & \text{if } a_i \leq s \leq b_i, \\ \frac{N}{\delta_i}(s - a_i) + N & \text{if } a_i - \delta_i \leq s \leq a_i, \\ -\frac{N}{\delta_i}(s - b_i) + N & \text{if } b_i \leq s \leq b_i + \delta_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f = \sum_{i=1}^m f_i$ is continuous and $\|f\|_{2,\lambda} = O(\frac{1}{N^3})$. Now consider $g = \frac{f}{\|f\|_{2,\lambda}}$. Inductively construct an orthonormal sequence in $C[0, 1]$ with the first function as g , orthogonal with respect to the λ measure. It is now clear that in this way the supremum in Eq. (2.2) can be made to exceed any large number.

Consequently, it follows that for all $t \in F^c$,

$$(2.3) \quad \tilde{\eta}_x^t \ll \lambda.$$

Finally from Lemma 3.6 of [12], it follows that $\eta_x \ll \lambda \otimes \lambda$. □

Remark 2.5. Note that the proof of Prop. 2.4 actually shows that $\tilde{\eta}_x^t \ll \lambda$ with $\frac{d\tilde{\eta}_x^t}{d\lambda} \in L^2([0, 1], \lambda)$ for λ almost all t .

The set of finite signed measures on the measurable space (X, σ_X) is a Banach space equipped with the total variation norm $\|\cdot\|_{t.v}$, also called the L_1 -norm, which is defined by $\|\mu\|_{t.v} = |\mu|(X)$, where $|\mu|$ denotes the variation measure of μ . It is well known that for probability measures μ and ν ,

$$(2.4) \quad \|\mu - \nu\|_{t.v} = 2 \sup_{B \in \sigma_X} |\mu(B) - \nu(B)| = \int_X |f - g| d\gamma$$

where f, g are density functions of μ, ν respectively with respect to any σ -finite measure γ dominating both μ, ν (see for instance Eq. (1.1) of [16]). We are now in a position to prove Thm. 2.1.

Proof of Thm. 2.1. Fix a set $S \subset \mathcal{M}$ such that $\mathbb{E}_A(x) = 0$ for all $x \in S$, $\overline{\text{span } S}^{\|\cdot\|_2} = L^2(A)^\perp$ and

$$\sum_{k=1}^{\infty} \|\mathbb{E}_A(xu_kx^*)\|_2^2 < \infty$$

for all $x \in S$, where $\{u_k\}_{k=1}^{\infty} \subset C[0,1]$ is an orthonormal basis of $L^2(A)$. There is a vector $\zeta \in L^2(A)^\perp$ such that $\|\zeta\|_2 = 1$ and $\eta_\zeta = \eta$. Choose a sequence $x_n \in \text{span } S$ such that $\|x_n\|_2 = 1$ and $x_n \rightarrow \zeta$ in $\|\cdot\|_2$ as $n \rightarrow \infty$. Then (see Lemma 3.10 [12]), we have $\eta_{x_n} \rightarrow \eta_\zeta = \eta$ in $\|\cdot\|_{t,v}$. Write $x_n = \sum_{i=1}^{k_n} c_{i,n} y_{i,n}$ with $y_{i,n} \in S$, $c_{i,n} \in \mathbb{C}$ for all $1 \leq i \leq k_n$ and $n \in \mathbb{N}$. As $y_{i,n} \in S$, so for all $n \in \mathbb{N}$ and $1 \leq i \leq k_n$,

$$\sum_{k=1}^{\infty} \|\mathbb{E}_A(y_{i,n}u_ky_{i,n}^*)\|_2^2 < \infty.$$

From Prop. 2.4 we have $\eta_{y_{i,n}} \ll \lambda \otimes \lambda$. But

$$\eta_{x_n} = \sum_{i=1}^{k_n} |c_{i,n}|^2 \eta_{y_{i,n}} + \sum_{i \neq j=1}^{k_n} c_{i,n} \bar{c}_{j,n} \eta_{y_{i,n}, y_{j,n}}.$$

For $1 \leq i \neq j \leq k_n$, the measures $\eta_{y_{i,n}, y_{j,n}}$ are possibly complex measures, but from Eq. (1.4), $|\eta_{y_{i,n}, y_{j,n}}| \leq \eta_{y_{i,n}} + \eta_{y_{j,n}} \ll \lambda \otimes \lambda$. Therefore $\eta_{x_n} \ll \lambda \otimes \lambda$. Since η_{x_n} is a probability measure so from Eq. (2.4),

$$\frac{1}{2} \|\eta_{x_n} - \eta_{x_m}\|_{t,v} = \int_{[0,1] \times [0,1]} |f_n(t,s) - f_m(t,s)| d(\lambda \otimes \lambda)(t,s) \rightarrow 0$$

as $n, m \rightarrow \infty$, where $f_n = \frac{d\eta_{x_n}}{d(\lambda \otimes \lambda)}$. Thus there is a function $f \in L^1([0,1] \times [0,1], \lambda \otimes \lambda)$ such that

$$\int_{[0,1] \times [0,1]} |f_n(t,s) - f(t,s)| d(\lambda \otimes \lambda)(t,s) \rightarrow 0$$

as $n \rightarrow \infty$. As η_{x_n} is a probability measure for each n , so $\|f_n\|_{L^1(\lambda \otimes \lambda)} = 1$ for all n . Therefore $\|f\|_{L^1(\lambda \otimes \lambda)} = 1$ and $\eta_{x_n} \rightarrow f d(\lambda \otimes \lambda)$ in $\|\cdot\|_{t,v}$. By uniqueness of limits $\eta = f d(\lambda \otimes \lambda)$.

We will now use condition (iii) of Thm. 2.1 to show that $\lambda \otimes \lambda \ll \eta$. Let $v \in A$ be the Haar unitary corresponding to the function $t \mapsto e^{2\pi i t}$. As noted after the statement of Thm. 2.1, we can assume $u = v$ in statement (iii) of Thm. 2.1. There is a nonzero vector $\xi_0 \in L^2(\mathcal{M}) \ominus L^2(A)$ such that $\mathbb{E}_A(\xi_0 v^n \xi_0^*) = 0$ for all $n \neq 0$. By 3° of Lemma 2.2 we have

$$\eta_{\xi_0}^t(1 \otimes v^n) = 0 \text{ for all } n \neq 0 \text{ and for } \lambda \text{ almost all } t.$$

Thus, the Fourier coefficients of the measure $\tilde{\eta}_{\xi_0}^t$ are $\tilde{\eta}_{\xi_0}^t(n) = 0$, $n \neq 0$ for λ almost all t . Thus $\tilde{\eta}_{\xi_0}^t$ is equal to a multiple of λ , for λ almost all t . The scalar above is the total mass of the measure and in this case is $\mathbb{E}_A(\xi_0 \xi_0^*)(t)$ for λ almost all t (see Lemma 2.2). A straight forward calculation shows that

$$\eta_{\xi_0}(a \otimes b) = \int_{[0,1] \times [0,1]} a(t)b(s) \mathbb{E}_A(\xi_0 \xi_0^*)(t) d(\lambda \otimes \lambda)(t,s), \quad a, b \in C[0,1].$$

Thus $\frac{d\eta_{\xi_0}}{d(\lambda \otimes \lambda)} = \mathbb{E}_A(\xi_0 \xi_0^*) \otimes 1$. Using Defn. 1.4 it is obvious that $\lambda \otimes \lambda \ll \eta_{\xi_0}$ as well. Thus $[\lambda \otimes \lambda] = [\eta_{\xi_0}]$.

Note that $\overline{A\xi_0 A}^{\|\cdot\|_2} \subseteq L^2(\mathcal{M}) \ominus L^2(A)$ and $\overline{A\xi_0 A}^{\|\cdot\|_2} \cong \int_{[0,1] \times [0,1]}^{\oplus} \mathbb{C}_{t,s} d\eta_{\xi_0}$, where $\mathbb{C}_{t,s} = \mathbb{C}$ and associated statement about diagonalizability of A holds. There are two cases to consider. If $\overline{A\xi_0 A}^{\|\cdot\|_2} = L^2(\mathcal{M}) \ominus L^2(A)$, then η_{ξ_0} is indeed the *left-right measure* of A . In this case there is nothing to prove. If there is a nonzero vector $\xi_1 \in L^2(\mathcal{M}) \ominus L^2(A) \ominus \overline{A\xi_0 A}^{\|\cdot\|_2}$, then by Lemma 5.7 [8]

$$\begin{aligned} & \text{either } \overline{A\xi_0 A}^{\|\cdot\|_2} \oplus \overline{A\xi_1 A}^{\|\cdot\|_2} \cong \int_{[0,1] \times [0,1]}^{\oplus} \mathcal{H}_{t,s} d(\eta_{\xi_0} + \nu), \quad 0 \neq \nu \perp \eta_{\xi_0}, \\ & \text{or } \overline{A\xi_0 A}^{\|\cdot\|_2} \oplus \overline{A\xi_1 A}^{\|\cdot\|_2} \cong \int_{[0,1] \times [0,1]}^{\oplus} \mathcal{H}_{t,s} d\eta_{\xi_0}, \end{aligned}$$

for a $\eta_{\xi_0} + \nu$ (or η_{ξ_0}) measurable field of Hilbert spaces $\mathcal{H}_{t,s}$, along with associated statement about the diagonalizable algebra. In the first case, use direct integrals to conclude that there is a nonzero vector $\tilde{\xi}_1 \in L^2(\mathcal{M})$ (which is off course orthogonal to $L^2(A) \oplus \overline{A\xi_0 A}^{\|\cdot\|_2}$) such that $\eta_{\tilde{\xi}_1} = \nu$. But by an argument analogous to the first part of the proof it follows that $\nu \ll \lambda \otimes \lambda$. This forces that $\overline{A\xi_0 A}^{\|\cdot\|_2} \oplus \overline{A\xi_1 A}^{\|\cdot\|_2} \cong \int_{[0,1] \times [0,1]}^{\oplus} \mathcal{H}_{t,s} d\eta_{\xi_0}$. Since we have to repeat this argument at most countably many times to exhaust $L^2(\mathcal{M}) \ominus L^2(A)$ (and in the process the multiplicity function will only change from Lemma 5.7 [8]), the proof is complete. \square

Remark 2.6. The sum in (ii) of Thm. 2.1 is the square of the Hilbert Schmidt norm of the operator $\mathbb{E}_A(x \cdot x^*)$. This is precisely the reason the sum is independent of the choice of the orthonormal basis. Note that if $\sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(xv^n x^*)\|_2^2 < \infty$ and $\sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(yv^n y^*)\|_2^2 < \infty$, where $x, y \in \mathcal{M}$, $\mathbb{E}_A(x) = 0$, $\mathbb{E}_A(y) = 0$ and v is the standard Haar unitary generator of A , then $\sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(xv^n y^*)\|_2^2 < \infty$. To see this, one has to use the facts $\frac{d\eta_{x,y}}{d(\lambda \otimes \lambda)}, \frac{d\eta_x}{d(\lambda \otimes \lambda)}, \frac{d\eta_y}{d(\lambda \otimes \lambda)} \in L^2(\lambda \otimes \lambda)$. Thus conditions (i), (ii) in Thm. 2.1 can be strengthened as: There exists a set $D \subset \mathcal{M}$ such that $\mathbb{E}_A(x) = 0$ for all $x \in D$, the set D is dense in $L^2(A)^\perp$ and

$$\sum_{k=1}^{\infty} \|\mathbb{E}_A(x_1 v_k x_2^*)\|_2^2 < \infty \text{ for all } x_1, x_2 \in D,$$

for some orthonormal basis $\{v_k\} \subset C[0,1]$ of $L^2(A)$. One choice of D is span S .

When the *left-right-measure* of a masa is the class of product measure, the masa satisfies conditions very close to the ones described in Thm. 2.1. This is the content of the next theorem.

For $\mathbb{N}_\infty \ni n \in \text{Puk}(A)$, let $E_n \subseteq [0,1] \times [0,1] \setminus \Delta([0,1])$ denote the set where the multiplicity function in $m.m(A)$ takes the value n . It is well known that E_n is η -measurable. Then

$$(2.5) \quad L^2(\mathcal{M}) \ominus L^2(A) \cong \bigoplus_{n \in \text{Puk}(A)} L^2(E_n, \eta|_{E_n}) \otimes \mathbb{C}^n \cong \bigoplus_{n \in \text{Puk}(A)} \int_{[0,1] \times [0,1]}^{\oplus} \mathbb{C}_{t,s}^n d\eta|_{E_n}(t, s),$$

where $\mathbb{C}_{t,s}^n = \mathbb{C}^n$ for $(t, s) \in E_n$ when $n < \infty$, and $\mathbb{C}^\infty = \mathbb{C}_{t,s}^\infty = l_2(\mathbb{N})$. Under this decomposition one has

$$\mathcal{A}'(1 - e_A) \cong \bigoplus_{n \in \text{Puk}(A)} L^\infty(E_n, \eta|_{E_n}) \overline{\otimes} \mathcal{M}_n(\mathbb{C}),$$

where $\mathcal{M}_\infty(\mathbb{C})$ is to be interpreted as $\mathbf{B}(l_2(\mathbb{N}))$. Consequently, it follows that for $\mathbb{N}_\infty \ni n \in \text{Puk}(A)$ the projections $\chi_{E_n} \otimes 1_n$ lie in $\mathbf{Z}(\mathcal{A}') = \mathcal{A}$, where 1_n denotes the identity of $\mathcal{M}_n(\mathbb{C})$ if $n < \infty$ and $1_\infty = 1_{\mathbf{B}(l_2(\mathbb{N}))}$. For $n \in \text{Puk}(A)$ choose vectors $\zeta_i^{(n)}$, $1 \leq i \leq n$ ($1 \leq i < \infty$ if $n = \infty$) so that the projections $P_i^{(n)} : L^2(\mathcal{M}) \mapsto \overline{A\zeta_i^{(n)} A}^{\|\cdot\|_2}$ are mutually orthogonal, equivalent in \mathcal{A}' , and $\sum_{i=1}^n P_i^{(n)} = \chi_{E_n} \otimes 1_n$.

Theorem 2.7. *Let $A \subset \mathcal{M}$ be a masa. Let the left-right-measure of A be the class of product measure. Then there is a set $S \subset L^2(\mathcal{M}) \ominus L^2(A)$ such that span S is dense in $L^2(A)^\perp$,*

$$\sum_{n=1}^{\infty} \|\mathbb{E}_A(\zeta w_n \zeta^*)\|_2^2 < \infty \text{ for all } \zeta \in S$$

for some orthonormal basis $\{w_n\}_{n=1}^\infty \subset A$ of $L^2(A)$, and there is a nonzero $\xi \in L^2(\mathcal{M}) \ominus L^2(A)$ such that $\mathbb{E}_A(\xi v^n \xi^*) = 0$ for all $n \neq 0$, where v is a Haar unitary generator of A .

Proof. We will first consider the case $\text{Puk}(A) = \{1\}$. In this case,

$$L^2(\mathcal{M}) \ominus L^2(A) \cong L^2([0,1] \times [0,1] \setminus \Delta([0,1]), \lambda \otimes \lambda),$$

the left and the right actions of A being given by

$$(af)(t, s) = a(t)f(t, s), (fb)(t, s) = b(s)f(t, s)$$

where $f \in L^2(A)^\perp$ and $a, b \in A$.

Let $0 \neq \zeta \in L^2(A)^\perp$ be a continuous function. Then for $a, b \in C[0, 1]$,

$$\begin{aligned} \langle a\zeta b, \zeta \rangle_{L^2(\mathcal{M})} &= \langle a\zeta b, \zeta \rangle_{L^2(\lambda \otimes \lambda)} = \int_{[0,1] \times [0,1]} a(t)b(s)\zeta(t,s)\overline{\zeta(t,s)}d\lambda(t)d\lambda(s) \\ &= \int_{[0,1] \times [0,1]} a(t)b(s)|\zeta(t,s)|^2d\lambda(t)d\lambda(s). \end{aligned}$$

Therefore $\frac{d\eta_\zeta}{d(\lambda \otimes \lambda)} = |\zeta|^2$ which is bounded, in particular in $L^2(\lambda \otimes \lambda)$. We claim that $\mathbb{E}_A(\zeta b \zeta^*) \in L^2(A)$ for any $b \in C[0, 1]$. Fix $a \in C[0, 1]$. Then as τ extends to L^1 ,

$$\begin{aligned} (2.6) \quad \int_0^1 a(t)\mathbb{E}_A(\zeta b \zeta^*)(t)d\lambda(t) &= \tau(a\mathbb{E}_A(\zeta b \zeta^*)) = \tau(a\zeta b \zeta^*) = \int_{[0,1] \times [0,1]} a(t)b(s)d\eta_\zeta(t,s) \\ &= \int_{[0,1] \times [0,1]} a(t)b(s)|\zeta|^2(t,s)d\lambda(t)d\lambda(s) \\ &= \int_0^1 a(t)\lambda(|\zeta|^2(t, \cdot)b)d\lambda(t). \end{aligned}$$

Now consider the function $[0, 1] \ni t \mapsto \lambda(|\zeta|^2(t, \cdot)b)$. It is clearly λ -measurable and

$$\begin{aligned} \int_0^1 \left| \lambda(|\zeta|^2(t, \cdot)b) \right|^2 d\lambda(t) &= \int_0^1 \left| \int_0^1 |\zeta|^2(t,s)b(s)d\lambda(s) \right|^2 d\lambda(t) \leq \|b\|^2 \int_0^1 \left(\int_0^1 |\zeta|^2(t,s)d\lambda(s) \right)^2 d\lambda(t) \\ &\leq \|b\|^2 \int_0^1 \int_0^1 |\zeta|^4(t,s)d\lambda(t)d\lambda(s) < \infty. \end{aligned}$$

Therefore from Eq. (2.6) we get,

$$\begin{aligned} \sup_{a \in C[0,1], \|a\|_2 \leq 1} \left| \int_0^1 a(t)\mathbb{E}_A(\zeta b \zeta^*)(t)d\lambda(t) \right| &= \sup_{a \in C[0,1], \|a\|_2 \leq 1} \left| \int_0^1 a(t)\lambda(|\zeta|^2(t, \cdot)b)d\lambda(t) \right| \\ &= \left(\int_0^1 \left| \lambda(|\zeta|^2(t, \cdot)b) \right|^2 d\lambda(t) \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

Consequently, it follows that $\mathbb{E}_A(\zeta b \zeta^*) \in L^2(A)$ and

$$\|\mathbb{E}_A(\zeta b \zeta^*)\|_2^2 = \int_0^1 \left| \lambda(|\zeta|^2(t, \cdot)b) \right|^2 d\lambda(t).$$

Let $v \in A$ be the Haar unitary corresponding to the function $t \mapsto e^{2\pi i t}$. Then $\{v^n\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $L^2(A)$ and by Parseval's theorem,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(\zeta v^n \zeta^*)\|_2^2 &= \sum_{n \in \mathbb{Z}} \int_0^1 \left| \lambda(|\zeta|^2(t, \cdot)v^n) \right|^2 d\lambda(t) = \int_0^1 \sum_{n \in \mathbb{Z}} \left| \lambda(|\zeta|^2(t, \cdot)v^n) \right|^2 d\lambda(t) \\ &= \int_0^1 \int_0^1 |\zeta|^4(t,s)d\lambda(s)d\lambda(t) < \infty. \end{aligned}$$

Thus $\{\zeta \in L^2(A)^\perp : \sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(\zeta v^n \zeta^*)\|_2^2 < \infty\}$ is dense in $L^2(A)^\perp$.

In the general case, write

$$L^2(\mathcal{M}) \ominus L^2(A) = \bigoplus_{n \in Puk(A)} \left(\bigoplus_{i=1}^n \overline{A \zeta_i^{(n)} A}^{\|\cdot\|_2} \right),$$

where $\zeta_i^{(n)}$ are vectors defined prior to the proof. For each $n \in Puk(A)$ and $1 \leq i \leq n$ (or $1 \leq i < n$ as the case may be), we consider the left and right actions of A on $\overline{A \zeta_i^{(n)} A}^{\|\cdot\|_2}$ to reduce the problem to a case similar to having one bicyclic vector. In this case, one works with bounded measurable functions.

Finally, let $\zeta \in L^2(\mathcal{M})$ correspond to the function $\chi_{\{(t,s): t \neq s\}}$. Then $\eta_\zeta = \lambda \otimes \lambda$. By arguments

exactly similar to the first part of the proof, conclude that $\mathbb{E}_A(\zeta a \zeta^*) \in L^2(A)$ for all $a \in A$. But by 3° of Lemma 2.2 we get,

$$\|\mathbb{E}_A(\zeta v^n \zeta^*)\|_2^2 = \int_0^1 |\eta_\zeta^t(1 \otimes v^n)|^2 d\lambda(t) = 0 \text{ for all } n \neq 0.$$

□

The proof of Thm. 2.1 and Thm. 2.7 yield the following corollary.

Corollary 2.8. *For a masa $A \subset \mathcal{M}$, the left-right-measure of A contains the product class as a summand, if and only if, there is a nonzero $\xi \in L^2(\mathcal{M}) \ominus L^2(A)$ such that $\mathbb{E}_A(\xi v^n \xi^*) = 0$ for all $n \neq 0$, where v is a Haar unitary generator of A .*

Proof. \Rightarrow By Lemma 5.7 [8], the left-right-measure of A is of the form $[\lambda \otimes \lambda + \nu]$ where either $\nu = 0$ or $\nu \perp \lambda \otimes \lambda$. In any case, there is a nonzero vector $\xi \in L^2(A)^\perp$ such that $\eta_\xi = \lambda \otimes \lambda$. Now use the argument of last part of Thm. 2.7. The reverse direction follows from the proof of Thm. 2.1. □

Remark 2.9. The proof of the previous theorem shows that if $A \subset \mathcal{M}$ is a masa satisfying the conditions of Thm. 2.1 or the left-right-measure of A is the product class, then there is a measurable partition $\{E_n\}_{n \in \text{Puk}(A)}$ of $\Delta([0, 1])^c$ such that

$${}_A L^2(\mathcal{M}) \ominus L^2(A)_A \cong \bigoplus_{n \in \text{Puk}(A)} \bigoplus_{i=1}^n {}_A L^2(E_n, \lambda \otimes \lambda)_A$$

with the natural actions on the right hand side.

Note that the *measure-multiplicity invariant* can be defined for any diffuse abelian subalgebra of \mathcal{M} in exactly the similar way defined in Defn. 1.3. If the diffuse abelian algebra is not a masa, then the diagonal will correspond to the L^2 completion of the relative commutant of the abelian algebra, so the multiplicity function along the diagonal will not be constantly 1. All other properties of the invariant will remain the same. We will use this observation in the following result.

Theorem 2.10. *Let $A \subset \mathcal{M}$ be a masa such that the left-right-measure of A is the class of product measure. Then for any diffuse algebra $B \subset A$, the left right-measure of B restricted to the off-diagonal is the class of product measure and $N(B)'' = B' \cap \mathcal{M} = A$.*

Proof. Since the left-right-measure of A is $[\lambda \otimes \lambda]$, so by Thm. 2.7, there is a set S orthogonal to $L^2(A)$, such that span S is dense in $L^2(A)^\perp$, $\sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(\zeta v^n \zeta^*)\|_2^2 < \infty$ for all $\zeta \in S$, where v is the standard Haar unitary generator of A . Moreover, the proof of Thm. 2.7 shows that we can assume $\frac{d\eta_\zeta}{d(\lambda \otimes \lambda)}$ is bounded $\lambda \otimes \lambda$ almost everywhere.

Arguments similar to the proof of Thm. 2.7 show that $\mathbb{E}_A(\zeta \cdot \zeta^*)$ defines a Hilbert Schmidt operator on $L^2(A)$. Fix a diffuse subalgebra $B \subset A$. Let $w \in B$ be a Haar unitary generator of B . Since $\mathbb{E}_A(\zeta \cdot \zeta^*)$ is Hilbert Schmidt, so $\sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(\zeta w^n \zeta^*)\|_2^2 < \infty$ and since $\|\mathbb{E}_B(\cdot)\|_2 \leq \|\mathbb{E}_A(\cdot)\|_2$ so

$$\sum_{n \in \mathbb{Z}} \|\mathbb{E}_B(\zeta w^n \zeta^*)\|_2^2 < \infty, \text{ for all } \zeta \in S.$$

Assuming $B = L^\infty([0, 1], \lambda)$ where λ is Lebesgue measure and using arguments required to prove Prop. 2.4, one finds $\eta_{\zeta, B} \ll \lambda \otimes \lambda$ for all $\zeta \in S$. The extra suffix refers to the fact that we are considering measures with respect to B . (It should be noted that the proof of Prop. 2.4 nowhere uses the fact that A is a masa.) Thus $\mathbb{E}_{B' \cap \mathcal{M}}(\zeta) = 0$ for all $\zeta \in S$. Indeed, write $\zeta = \zeta_1 + \zeta_2$ with $\mathbb{E}_{B' \cap \mathcal{M}}(\zeta) = \zeta_1$ and $\mathbb{E}_{B' \cap \mathcal{M}}(\zeta_2) = 0$. For $a, b \in B$ one has

$$\langle a \zeta_1 b, \zeta_2 \rangle = \tau(a \zeta_1 b \zeta_2^*) = \tau(\mathbb{E}_{B' \cap \mathcal{M}}(a \zeta_1 b) \zeta_2^*) = 0.$$

Thus $\eta_{\zeta, B} = \eta_{\zeta_1, B} + \eta_{\zeta_2, B}$. But $\eta_{\zeta_1, B} \ll \tilde{\Delta}_* \lambda$ (for $\tilde{\Delta}$ see §1) with the Radon-Nikodym derivative given by $\mathbb{E}_B(\zeta_1 \zeta_1^*)$. Consequently, $\zeta_1 = 0$. Thus $S \subset L^2(B' \cap \mathcal{M})^\perp$ and hence $L^2(A)^\perp \subseteq L^2(B' \cap \mathcal{M})^\perp$. It follows that $B' \cap \mathcal{M} = A$.

By arguments similar to the proof of Thm. 2.1, it follows that any member in left-right-measure of B restricted to the off-diagonal is dominated by $\lambda \otimes \lambda$.

There is a vector $0 \neq \xi \in L^2(A)^\perp$ such that $\mathbb{E}_A(\xi v^n \xi^*) = 0$ for all $n \neq 0$. It follows that $\mathbb{E}_A(\xi a \xi^*) = 0$ for all $a \in A$ with $\tau(a) = 0$. Consequently, $\mathbb{E}_A(\xi w^n \xi^*) = 0$ and hence $\mathbb{E}_B(\xi w^n \xi^*) = 0$ for all $n \neq 0$. By arguments made in the last part of the proof of Thm. 2.1, it follows that the left-right-measure of B restricted to the off diagonal is the class of product measure.

Finally, if $0 \neq \zeta_0 \in L^2(N(B)'')$, then $\overline{B\zeta_0 B}^{\|\cdot\|_2} \in C_d(B)$ (Prop. 3.11 [12]). Thus by using Lemma 5.7 [8], it follows $\eta_{\zeta_0, B}$ must be supported on the diagonal; equivalently $\mathbb{E}_{B' \cap \mathcal{M}}(\zeta_0) = \zeta_0$. Thus $\zeta_0 \in L^2(A)$. This completes the proof. \square

3. TAUER MASAS IN THE HYPERFINITE II_1 FACTOR

In this section, we will calculate the *left-right-measures* of certain Tauer masas in the hyperfinite II_1 factor \mathcal{R} . The examples of Tauer masas in which we are interested are directly taken from [23].

Definition 3.1. (White) A masa A in \mathcal{R} is said to be a *Tauer masa*, if there exists a sequence of finite type I subfactors $\{\mathcal{N}_n\}_{n=1}^\infty$ such that,

- (i) $\mathcal{N}_n \subset \mathcal{N}_{n+1}$ for all n ,
- (ii) $(\bigcup_{n=1}^\infty \mathcal{N}_n)'' = \mathcal{R}$,
- (iii) $A_n = A \cap \mathcal{N}_n$ is a masa in \mathcal{N}_n for every n .

This allows one to write the structure of every Tauer masa A in \mathcal{R} with respect to the chain $\{\mathcal{N}_n\}_{n=1}^\infty$ as follows. Switching to the notation of tensor products, the above definition means that we can find finite type I subfactors $\{\mathcal{M}_n\}_{n=1}^\infty$ such that, $\mathcal{N}_n = \bigotimes_{r=1}^n \mathcal{M}_r$ for every n . For $m > n$, the m -th finite dimensional approximation of A can be written in terms of the n -th one as,

$$(3.1) \quad A_m = \bigoplus_{e \in \mathcal{P}(A_n)} e \otimes A_{m,n}^{(e)},$$

where the direct sum is over the set of minimal projections $\mathcal{P}(A_n)$ in A_n and $A_{m,n}^{(e)}$ is a masa in $\bigotimes_{r=n+1}^m \mathcal{M}_r$. Note that the Cartan masa arising from the infinite tensor product of diagonal matrices inside the hyperfinite II_1 factor is a Tauer masa. In Thm. 4.1 [32], White had shown that the *Pukánszky invariant* of every Tauer masa is $\{1\}$. In fact, it follows from his proof that the bicyclic vector for any Tauer masa can be chosen to be an operator from \mathcal{R} itself.

Sinclair and White [23] has exhibited a continuous path of singular masas in \mathcal{R} , no two of which can be connected by automorphisms of \mathcal{R} . We are interested in two masas that correspond to the end points of this path. For all Tauer masas, it is clear that the Cantor set is the natural space where we have to build the measures. For ease of calculation, we need to index the minimal projections in the approximating stages in an appropriate fashion. It is now time to introduce some notation.

1° Notation : If $\mathcal{N}_n = \bigotimes_{r=1}^n \mathcal{M}_{k_r}(\mathbb{C})$, then the minimal projections of A_n will be denoted by $^{(n)}f_{\underline{t}(n)}$, where $\underline{t}(n) = (t_1, t_2, \dots, t_n)$ with $1 \leq t_i \leq k_i$, $1 \leq i \leq n$. The convention that we follow is

$$^{(n)}f_{(t_1, t_2, \dots, t_n)} = {}^{(n-1)}f_{(t_1, t_2, \dots, t_{n-1})} \otimes {}^{(n)}e_{t_n}^{(t_1, t_2, \dots, t_{n-1})},$$

where $^{(n)}e_{t_n}^{(t_1, t_2, \dots, t_{n-1})}$ are the minimal projections of the algebra $A_{n, n-1}^{(t_1, t_2, \dots, t_{n-1})}$, in accordance with Eq. (3.1). The matrix units corresponding to this family of minimal projections will be denoted by $^{(n)}f_{\underline{t}(n), \underline{s}(n)}$ and we will understand $^{(n)}f_{\underline{t}(n), \underline{t}(n)} = {}^{(n)}f_{\underline{t}(n)}$. For two tuples (t_1, t_2, \dots, t_n) and (s_1, s_2, \dots, s_n) such that $t_i = s_i$ for $1 \leq i \leq n-1$ and $t_n \neq s_n$, we will write $^{(n)}f_{\underline{t}(n), \underline{s}(n)} = {}^{(n)}f_{(\cdot, t_n), (\cdot, s_n)}$.

2° Notation: For any two subsets $S, T \subseteq \mathcal{M}$, we will denote by $S \cdot T$ the set $\text{span}\{ab : a \in S, b \in T\}$. The normalized trace of $\mathcal{M}_n(\mathbb{C})$ will be denoted by tr_n . The unique normal tracial state of the hyperfinite factor \mathcal{R} will be denoted by $\tau_{\mathcal{R}}$. This trace $\tau_{\mathcal{R}}$ when restricted to A gives rise to a measure on a Cantor set which will also be denoted by $\tau_{\mathcal{R}}$.

Recall from [19] that two subalgebras B, C in a finite factor N are called orthogonal with respect to the unique normal tracial state τ_N , if $\tau_N(bc) = \tau_N(b)\tau_N(c)$ for all $b \in B, c \in C$. The next lemma is very well known but we record it for convenience.

Lemma 3.2. *If A, B are two masas in $\mathcal{M}_n(\mathbb{C})$ orthogonal with respect to the normalized trace tr_n , then $A \cdot B = \mathcal{M}_n(\mathbb{C})$.*

3.1. Tauer Masa of Product Class.

Following Sinclair and White [23], we shall calculate the *measure-multiplicity invariant* of a Tauer masa A , whose description is elaborated below. The Γ invariant of this Tauer masa is 0 (A is totally non- Γ [23]). We will show that its *left-right-measure* belongs to the product class. This example is important, as, it is an example of a masa in \mathcal{R} with simple multiplicity whose *left-right-measure* is the class of product measure. Such masas are rare in \mathcal{R} . We do not know whether it arises from a dynamical system.

Let $k_1 = 2$, and for each $r \geq 2$, let k_r be a prime exceeding $k_1 k_2 \cdots k_{r-1}$. Set \mathcal{M}_r to be the algebra of $k_r \times k_r$ matrices. By Thm. 3.2 [19], there is a family $\{^{(r)}D^{\underline{t}(r-1)}\}_{\underline{t}(r-1)}$ of pairwise orthogonal masas in \mathcal{M}_r . Let $\mathcal{N}_n = \bigotimes_{r=1}^n \mathcal{M}_r$. There is a natural inclusion $x \mapsto x \otimes 1$ of \mathcal{N}_n inside \mathcal{N}_{n+1} and one works in the hyperfinite II_1 factor \mathcal{R} , obtained as a direct limit of these \mathcal{N}_n with respect to the normalized trace. With respect to the chain $\{\mathcal{N}_n\}_{n=1}^\infty$ of finite type I subfactors of \mathcal{R} , the masa A is constructed as follows.

Let $A_1 = D_2(\mathbb{C}) \subset \mathcal{M}_1$ be the diagonal masa. Having constructed A_n , one constructs A_{n+1} as,

$$(3.2) \quad A_{n+1} = \bigoplus_{\underline{t}(n)}^{(n)} f_{\underline{t}(n)} \otimes^{(n+1)} D^{\underline{t}(n)}.$$

That $(\cup_{n=1}^\infty A_n)''$ is a masa in \mathcal{R} , follows from a theorem of Tauer (see Thm. 2.5 [29]). This Tauer masa is singular from Prop. 2.1 [23].

We denote by $P_{\underline{t}(n), \underline{s}(n)}^{(n)}$ the orthogonal projection from $L^2(\mathcal{R})$ onto the subspace $^{(n)}f_{\underline{t}(n)} L^2(\mathcal{R})^{(n)} f_{\underline{s}(n)}$, and let,

$$(3.3) \quad P = \sum_{n=1}^\infty \sum_{\substack{\underline{t}(n), \underline{s}(n): \\ t_1=s_1, \dots, t_{n-1}=s_{n-1}, t_n \neq s_n}} P_{(\underline{t}(n), \dots, \underline{t}(n)), (\underline{s}(n), \dots, \underline{s}(n))}^{(n)}.$$

Clearly, $P_{\underline{t}(n), \underline{s}(n)}^{(n)} = ^{(n)}f_{\underline{t}(n)} J ^{(n)}f_{\underline{s}(n)} J$ and is in \mathcal{A} . At the first sight, it might not be clear that the sum in Eq. (3.3) makes sense, but, the projections involved in the sum are orthogonal and sums to $1 - e_A$. Indeed, since e_A is the limit in strong operator topology of $e_{A'_n \cap \mathcal{R}} = \sum_{\underline{t}(n)} P_{(\underline{t}(n), \dots, \underline{t}(n)), (\underline{t}(n), \dots, \underline{t}(n))}^{(n)}$ (§5.3 [24], Lemma 1.2 [17]), that $P = 1 - e_A$ follows by rearranging terms in Eq. (3.3).

The following lemma, part of which was recorded by Sinclair and White [23], will be crucial for our calculations.

Lemma 3.3. *For each $n \in \mathbb{N}$, let $\mathcal{R} = \mathcal{N}_n \otimes \mathcal{R}_n$, where $\mathcal{R}_n = (\bigotimes_{r=n+1}^\infty \mathcal{M}_{k_r}(\mathbb{C}))''$. Then*

$$(3.4) \quad A = \bigoplus_{\underline{t}(n)}^{(n)} f_{\underline{t}(n)} \otimes A_{\infty, n+1}^{\underline{t}(n)}, \text{ where}$$

$A_{\infty, n+1}^{\underline{t}(n)}$ are Tauer masas in \mathcal{R}_n and whenever $\underline{t}(n) \neq \underline{s}(n)$ we have

(i) $A_{\infty, n+1}^{\underline{t}(n)}$ and $A_{\infty, n+1}^{\underline{s}(n)}$ are orthogonal in \mathcal{R}_n ,

(ii) $(A_{\infty, n+1}^{\underline{t}(n)} \cdot A_{\infty, n+1}^{\underline{s}(n)})^{-\|\cdot\|_2} = L^2(\mathcal{R}_n)$.

Moreover, for each $\underline{t}(n)$ if $\{A_{m, n+1}^{\underline{t}(n)}\}_{m=1}^\infty$ denote the m -th approximation of $A_{\infty, n+1}^{\underline{t}(n)}$ in \mathcal{R}_n , then

$$(3.5) \quad A_{1, n+1}^{\underline{t}(n)} = ^{(n+1)}D^{\underline{t}(n)}, \text{ and,}$$

$$(3.6) \quad A_{m+1, n+1}^{\underline{t}(n)} = \bigoplus_{e \in \mathcal{P}(A_{m, n+1}^{\underline{t}(n)})} e \otimes^{(m+1)} D_{e, n+1}^{\underline{t}(n)},$$

where for each fixed m and $\underline{t}(n)$, the family $\{^{(m+1)}D_{e, n+1}^{\underline{t}(n)}\}_e$ are pairwise orthogonal masas in $\mathcal{M}_{k_{n+m+1}}(\mathbb{C})$.

Proof. It should be understood that in (ii) of the statement, the closure is taken with respect to the faithful normal tracial state of \mathcal{R}_n . We only have to prove (ii). The rest of the statements are just

rephrasing of Lemma 5.6 of [23].

Use Lemma 3.2, (i) and Eq. (3.5) to conclude that

$$\mathcal{M}_{k_{n+1}} \subseteq (A_{\infty, n+1}^{\underline{t}(n)} \cdot A_{\infty, n+1}^{\underline{s}(n)})^{-\|\cdot\|_2}.$$

Since $A_{\infty, n+1}^{\underline{t}(n)}$ and $A_{\infty, n+1}^{\underline{s}(n)}$ are orthogonal, so is $A_{m, n+1}^{\underline{t}(n)}$ and $A_{m, n+1}^{\underline{s}(n)}$ for all $m \geq n+1$. Use Lemma 3.2 to conclude that $\bigotimes_{r=n+1}^m \mathcal{M}_{k_r}(\mathbb{C}) \subseteq (A_{\infty, n+1}^{\underline{t}(n)} \cdot A_{\infty, n+1}^{\underline{s}(n)})^{-\|\cdot\|_2}$ for all $m \geq n+1$. Now use density of algebraic tensor product of matrix algebras in $L^2(\mathcal{R}_n)$ to finish the proof. \square

For each n , let $X_n = \{x_1^{(n)}, x_2^{(n)}, \dots, x_{k_n}^{(n)}\}$ denote a set of k_n points. Let $Y^{(n)} = \prod_{k=1}^n X_k$, $X^{(n)} = \prod_{k=n+1}^{\infty} X_k$ and $X = \prod_{k=1}^{\infty} X_k$, so that for each n , $X = Y^{(n)} \times X^{(n)}$. Therefore, $X = \lim_{\infty \leftarrow n} Y^{(n)}$ and $C(X)$ is norm separable and *w.o.t* dense in A . The identification is a standard one and we omit the details. Write $B = C(X)$. Therefore,

$$(3.7) \quad B = \bigoplus_{\underline{t}(n)}^{(n)} f_{\underline{t}(n)} \otimes B_{\infty, n+1}^{\underline{t}(n)},$$

$B_{\infty, n+1}^{\underline{t}(n)} \cong C(X^{(n+1)})$ and is a *w.o.t* dense, norm separable C^* subalgebra of $A_{\infty, n+1}^{\underline{t}(n)}$.

Lemma 3.4. For each n and $\underline{t}(n) \neq \underline{s}(n)$,

(i) $(A^{(n)} f_{\underline{t}(n), \underline{s}(n)} A)^{-\|\cdot\|_2} = {}^{(n)} f_{\underline{t}(n)} L^2(\mathcal{R})^{(n)} f_{\underline{s}(n)}$,

(ii) for $a, b \in B$,

$$\langle a^{(n)} f_{\underline{t}(n), \underline{s}(n)} b, {}^{(n)} f_{\underline{t}(n), \underline{s}(n)} \rangle_{\tau_{\mathcal{R}}} = k_1 k_2 \cdots k_n \int_X \int_X {}^{(n)} f_{\underline{t}(n)}(t) {}^{(n)} f_{\underline{s}(n)}(s) a(t) b(s) d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s).$$

Moreover, $(A^{(n)} f_{\underline{t}(n), \underline{s}(n)} A)^{-\|\cdot\|_2}$ is orthogonal to $(A^{(n)} f_{\underline{t}'(n), \underline{s}'(n)} A)^{-\|\cdot\|_2}$ whenever $\underline{t}(n) \neq \underline{s}(n)$, $\underline{t}'(n) \neq \underline{s}'(n)$ and $(\underline{t}(n), \underline{s}(n)) \neq (\underline{t}'(n), \underline{s}'(n))$.

Proof. For $a, b \in A$, using Eq. (3.4) write

$$a = \bigoplus_{\underline{q}(n)}^{(n)} f_{\underline{q}(n)} \otimes a_{\underline{q}(n)} \text{ and } b = \bigoplus_{\underline{p}(n)}^{(n)} f_{\underline{p}(n)} \otimes b_{\underline{p}(n)}$$

for $a_{\underline{q}(n)} \in A_{\infty, n+1}^{\underline{q}(n)}$, and $b_{\underline{p}(n)} \in A_{\infty, n+1}^{\underline{p}(n)}$. By direct multiplication, we get

$$a^{(n)} f_{\underline{t}(n), \underline{s}(n)} \otimes 1_{\mathcal{R}_n} b = {}^{(n)} f_{\underline{t}(n), \underline{s}(n)} \otimes a_{\underline{t}(n)} b_{\underline{s}(n)}.$$

Therefore (i) follows from (ii) of Lemma 3.3. Moreover, for $a, b \in B$,

$$\begin{aligned} & \langle a^{(n)} f_{\underline{t}(n), \underline{s}(n)} \otimes 1_{\mathcal{R}_n} b, {}^{(n)} f_{\underline{t}(n), \underline{s}(n)} \otimes 1_{\mathcal{R}_n} \rangle_{\tau_{\mathcal{R}}} \\ &= \text{tr}_{\prod_{i=1}^n k_i} ({}^{(n)} f_{\underline{t}(n)} \tau_{\mathcal{R}_n} (a_{\underline{t}(n)} b_{\underline{s}(n)})) \\ &= \frac{1}{k_1 k_2 \cdots k_n} \tau_{\mathcal{R}_n} (a_{\underline{t}(n)} b_{\underline{s}(n)}) \\ &= k_1 k_2 \cdots k_n \tau_{\mathcal{R}} (a({}^{(n)} f_{\underline{t}(n)} \otimes 1)) \tau_{\mathcal{R}} (b({}^{(n)} f_{\underline{s}(n)} \otimes 1)) \text{ (by orthogonality, Lemma 3.3 (ii))} \\ &= k_1 k_2 \cdots k_n \int_X \int_X a(t) ({}^{(n)} f_{\underline{t}(n)} \otimes 1)(t) b(s) ({}^{(n)} f_{\underline{s}(n)} \otimes 1)(s) d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s) \\ &= k_1 k_2 \cdots k_n \int_{x_{t_1}^{(1)} \times \cdots \times x_{t_n}^{(n)} \times X^{(n)}} \int_{x_{s_1}^{(1)} \times \cdots \times x_{s_n}^{(n)} \times X^{(n)}} a_{\underline{t}(n)}(t) b_{\underline{s}(n)}(s) d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s) \\ &= k_1 k_2 \cdots k_n \int_{X \times X} {}^{(n)} f_{\underline{t}(n)}(t) {}^{(n)} f_{\underline{s}(n)}(s) a(t) b(s) d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s), \end{aligned}$$

where the indicators of $(x_{t_1}^{(1)} \times \cdots \times x_{t_n}^{(n)}) \times X^{(n)}$ and $(x_{s_1}^{(1)} \times \cdots \times x_{s_n}^{(n)}) \times X^{(n)}$ corresponds to ${}^{(n)} f_{\underline{t}(n)}$ and ${}^{(n)} f_{\underline{s}(n)}$ respectively. This proves (ii). Clearly the final statement follows from (i) and the fact

that ${}^{(n)}f_{\underline{t}(n)}J^{(n)}f_{\underline{s}(n)}J$ and ${}^{(n)}f_{\underline{t}'(n)}J^{(n)}f_{\underline{s}'(n)}J$ are orthogonal projections in $L^2(\mathcal{R})$ if $(\underline{t}(n), \underline{s}(n)) \neq (\underline{t}'(n), \underline{s}'(n))$. \square

Remark 3.5. The following observation will be used in the next proof. On every occasion below, where we add direct integrals, Lemma 5.7 [8] is invoked. For $t_i = s_i$, $1 \leq i \leq n-1$ and $t_n \neq s_n$, the projection $P_{(t_1, \dots, t_n), (s_1, \dots, s_n)}^{(n)} \in \mathcal{A}'$ and hence is in \mathcal{A} , as \mathcal{A} is maximal abelian in $\mathbf{B}(L^2(\mathcal{R}))$. Therefore, $P_{(t_1, \dots, t_n), (s_1, \dots, s_n)}^{(n)}$ is decomposable (see Ch. 14 [10]). Denote

$$E_{(\cdot, t_n), (\cdot, s_n)} = (x_{t_1}^{(1)} \times \dots \times x_{t_{n-1}}^{(n-1)} \times x_{t_n}^{(n)} \times X^{(n)}) \times (x_{t_1}^{(1)} \times \dots \times x_{t_{n-1}}^{(n-1)} \times x_{s_n}^{(n)} \times X^{(n)}).$$

From Lemma 3.4, it follows that the range $P_{(t_1, \dots, t_n), (s_1, \dots, s_n)}^{(n)}(L^2(\mathcal{R}))$ is the direct integral of complex numbers over the set $E_{(\cdot, t_n), (\cdot, s_n)}$ with respect to $\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}$, and, $\mathcal{A}P_{(t_1, \dots, t_n), (s_1, \dots, s_n)}^{(n)}$ is the diagonalizable algebra with respect to this decomposition. For $\underline{t}(n-1) \neq \underline{t}'(n-1)$, the direct integrals of $P_{(\underline{t}(n-1), t_n), (\underline{t}(n-1), s_n)}^{(n)}$ and $P_{(\underline{t}'(n-1), t'_n), (\underline{t}'(n-1), s'_n)}^{(n)}$ with $t_n \neq s_n$ and $t'_n \neq s'_n$ rest over disjoint subsets of $X \times X$. Therefore, the range of $P^{(n)} = \sum_{\substack{\underline{t}(n), \underline{s}(n): \\ t_1=s_1, \dots, t_{n-1}=s_{n-1}, t_n \neq s_n}} P_{(t_1, \dots, t_n), (s_1, \dots, s_n)}^{(n)}$ is the direct integral of

complex numbers with respect to $\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}$ over the set $E_n = \cup_{t_1=1}^{k_1} \dots \cup_{t_{n-1}=1}^{k_{n-1}} \cup_{t_n \neq s_n=1}^{k_n} E_{(\cdot, t_n), (\cdot, s_n)}$, and associated statements about diagonalizability of $\mathcal{A}P^{(n)}$ hold. It is important to note that $E_n \cap E_m = \emptyset$ for all $n \neq m$.

Let $c_n = \prod_{r=1}^n k_r$ for $n \geq 1$ and $c_0 = 1$.

Proposition 3.6. *The vector $\sum_{n=1}^{\infty} \sum_{\underline{t}(n)} \frac{1}{\sqrt{c_n}} {}^{(n)}f_{(\cdot, t_n), (\cdot, s_n)}$ is a cyclic vector of $\mathcal{A}(1 - e_A)$ and*

$$(1 - e_A)(L^2(\mathcal{R})) \cong \int_{X \times X}^{\oplus} \mathbb{C}_{t,s} d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s), \text{ where } \mathbb{C}_{t,s} = \mathbb{C}.$$

Moreover, $\mathcal{A}(1 - e_A)$ is the algebra of diagonalizable operators with respect to this decomposition.

Proof. Fix $n \in \mathbb{N}$. For each $1 \leq t_i \leq k_i$, $1 \leq i \leq n-1$, and $1 \leq t_n \neq s_n \leq k_n$, working with vectors $\frac{1}{\sqrt{c_n}} {}^{(n)}f_{(\cdot, t_n), (\cdot, s_n)}$, one finds (using Lemma 3.4) a positive measure $\eta_{(t_1, \dots, t_n), (s_1, \dots, s_n)}^{(n)}$ supported on $E_{(\cdot, t_n), (\cdot, s_n)}$ such that

$$d\eta_{(t_1, \dots, t_n), (s_1, \dots, s_n)}^{(n)} = {}^{(n)}f_{(t_1, \dots, t_n)} \otimes {}^{(n)}f_{(s_1, \dots, s_n)} d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}).$$

By making arguments similar to Rem. 3.5, for each n find a positive measure $\eta^{(n)}$ on E_n such that $\eta^{(n)} = \chi_{E_n} d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})$ and

$$P^{(n)}(L^2(\mathcal{R})) = \sum_{\substack{\underline{t}(n), \underline{s}(n): \\ t_1=s_1, \dots, t_{n-1}=s_{n-1}, t_n \neq s_n}} P_{(t_1, \dots, t_n), (s_1, \dots, s_n)}^{(n)}(L^2(\mathcal{R})) \cong \int_{X \times X}^{\oplus} \mathbb{C}_{t,s} d\eta^{(n)}(t, s),$$

where $\mathbb{C}_{t,s} = \mathbb{C}$ and $\mathcal{A}P^{(n)}$ is diagonalizable with respect to this decomposition. Note that

$$(3.8) \quad \eta^{(n)}(X \times X) = \frac{c_{n-1}(k_n^2 - k_n)}{c_n^2} = \frac{1}{c_{n-1}} - \frac{1}{c_n}.$$

From Rem. 3.5, note that the measures $\eta^{(n)}$ are supported on disjoint sets. Hence by Lemma 5.7 [8],

$$(3.9) \quad (1 - e_A)(L^2(\mathcal{R})) \cong \int_{X \times X}^{\oplus} \mathbb{C}_{t,s} d\eta(t, s), \text{ where } \mathbb{C}_{t,s} = \mathbb{C}, \eta = \sum_{n=1}^{\infty} \eta^{(n)}.$$

Moreover, $\mathcal{A}(1 - e_A)$ is diagonalizable with respect to the decomposition in Eq. (3.9). Clearly,

$$\eta(X \times X) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \eta^{(n)}(X \times X) = \lim_{N \rightarrow \infty} \frac{1}{c_0} - \frac{1}{c_N} = 1.$$

Finally, $\eta = \tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}$. Indeed, for $a, b \in C(X)$,

$$\begin{aligned} & \int_{X \times X} a(t)b(s) d\eta(t, s) \\ &= \sum_{n=1}^{\infty} \int_{X \times X} a(t)b(s) d\eta^{(n)}(t, s) \\ &= \sum_{n=1}^{\infty} \sum_{\substack{\underline{t}(n), \underline{s}(n): \\ t_1=s_1, \dots, t_{n-1}=s_{n-1}, t_n \neq s_n}} \int_{X \times X} a(t)b(s)^{(n)} f_{(t_1, \dots, t_n)}(t)^{(n)} f_{(s_1, \dots, s_n)}(s) d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s). \end{aligned}$$

But $\sum_{n=1}^N \sum_{\substack{\underline{t}(n), \underline{s}(n): \\ t_1=s_1, \dots, t_{n-1}=s_{n-1}, t_n \neq s_n}}^{(n)} f_{(t_1, \dots, t_n)}(t)^{(n)} f_{(s_1, \dots, s_n)}(s) \uparrow \chi_{\Delta(X)^c}$ pointwise $\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}$ almost ev-

erywhere. Use dominated convergence theorem and the fact $(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(\Delta(X)) = 0$ to conclude $\eta = \tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}$. This completes the proof. \square

For A , the operator $x = \sum_{n=1}^{\infty} \sum_{\underline{t}(n)} \frac{1}{\sqrt{c_n}}^{(n)} f_{(\cdot, t_n), (\cdot, s_n)}$ gives rise to a choice of a vector in (iii) of

Thm. 2.1. In order to get an appropriate vector one has to apply an appropriate transformation between the Cantor set and $[0, 1]$, which will induce a unitary in $\mathbf{B}(L^2(\mathcal{R}))$ preserving the bimodule structure. Since the *Pukánszky invariant* of every *Tauer masa* is $\{1\}$, we have computed the *measure-multiplicity invariant* of A . Note that AxA is dense in $L^2(\mathcal{R}) \ominus L^2(A)$. For $a \in A$ and any orthonormal basis $\{v_n\}_{n=1}^{\infty} \subset A$ of $L^2(A)$, one has $\sum_n \|\mathbb{E}_A(xav_n x^*)\|_2^2 = \sum_n \int_X |\eta_x^t(1 \otimes av_n)|^2 d\tau_{\mathcal{R}}(t) = \|a\|_2^2$, as $\eta_x = \tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}$ (see Lemma 3.6 [12]). This shows that the Tauer masa above satisfy conditions (i) and (ii) of Thm. 2.1 with $S = AxA$. The above Tauer masa was denoted by $A(0)$ in [23]. There is a Tauer masa of exactly opposite flavor, which we call the *alternating Tauer masa*.

3.2. Alternating Tauer Masa.

The *alternating Tauer masa* $A(1)$ is a singular Tauer masa in the hyperfinite II_1 factor \mathcal{R} , constructed by White and Sinclair [23]. It contains nontrivial centralizing sequences of \mathcal{R} . In fact, its Γ -invariant is 1. This masa will play a role in §5. In §4, we will describe its *left-right-measure*.

The chain for this masa is exactly similar to the masa of the product class described before. Let $A(1)_1 = D_2(\mathbb{C}) \subset \mathcal{M}_1$ be the diagonal masa. Having constructed $A(1)_n \subset \mathcal{N}_n$, one constructs $A(1)_{n+1}$ as,

$$A(1)_{n+1} = \begin{cases} A(1)_n \otimes {}^{(n+1)}D_{n+1}, & n \text{ even, } {}^{(n+1)}D_{n+1} \text{ is the diagonal masa in } \mathcal{M}_{k_{n+1}}(\mathbb{C}), \\ \bigoplus_{\underline{t}(n)} {}^{(n)}f_{\underline{t}(n)} \otimes {}^{(n+1)}D^{\underline{t}(n)}, & n \text{ odd, } {}^{(n+1)}D^{\underline{t}(n)} \text{ pairwise orthogonal in } \mathcal{M}_{k_{n+1}}(\mathbb{C}). \end{cases}$$

We will prove that the *left-right-measure* of $A(1)$ is singular with respect to the product measure. Having understood the *left-right-measures* of $A(0)$ and $A(1)$, we can describe the same for the entire path of masas exhibited in [23].

4. Γ AND NON Γ MASAS

In this section, we study properties of *left-right-measures* of masas that possess nontrivial centralizing sequences of the factor. We also study properties of *left-right-measures* that prevent a masa to contain nontrivial centralizing sequences. This section contains partial answers. Some results in this section can be proved by bringing in the notion of strongly mixing masas [9]. To keep this paper in a reasonable size, we postpone the notion of strong mixing to a future paper.

Definition 4.1. A centralizing sequence in a II_1 factor \mathcal{M} is a bounded sequence $\{x_n\} \subset \mathcal{M}$ such that $\|x_n y - y x_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$ for all $y \in \mathcal{M}$. The centralizing sequence $\{x_n\}$ is trivial, if there exists a sequence $\lambda_n \in \mathbb{C}$ such that $\|x_n - \lambda_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$.

For a masa $A \subset \mathcal{M}$, the Γ invariant of A is defined by

$\Gamma(A) = \sup\{\tau(p) : p \in A \text{ is a projection and } Ap \text{ contains nontrivial centralizing sequences of } p\mathcal{M}p\}$. It is immediate that $\Gamma(A) = \Gamma(\theta(A))$, where θ is an automorphism of \mathcal{M} [23]. If $\Gamma(A) = 0$, then A is said to be totally non- Γ . We continue to assume that $A = L^\infty([0, 1], \lambda)$, where λ is the Lebesgue measure.

Proposition 4.2. *Let $A \subset \mathcal{M}$ be a masa. Let the left-right-measure of A be $[(\lambda \otimes \lambda) + \mu]$, where $\mu \perp \lambda \otimes \lambda$ and μ is finite. Then A cannot contain non trivial centralizing sequences of \mathcal{M} . Moreover, $\Gamma(A) = 0$.*

Proof. Write $[0, 1] \times [0, 1] \setminus \Delta([0, 1]) = E \cup F$, where $(\lambda \otimes \lambda)(E) = 0$ and $\mu(F) = 0$. There exists a nonzero vector $\zeta \in L^2(\mathcal{M}) \ominus L^2(A)$ such that for $a, b \in C[0, 1]$,

$$\eta_\zeta(a \otimes b) = \lambda(a)\lambda(b).$$

The direct integral of ζ is supported on F .

If possible, let $\{a_n\} \subset A$ be a non trivial centralizing sequence. By making a density argument, we can assume that $a_n = a_n^* \in C[0, 1]$ and $\tau(a_n) = 0$ for all n . Also assume that $\limsup_n \|a_n\|_2 = \alpha > 0$.

A triangle inequality argument shows that $\|a_n \zeta - \zeta a_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$. However,

$$(4.1) \quad \begin{aligned} \|a_n \zeta - \zeta a_n\|_2^2 &= \langle a_n \zeta, a_n \zeta \rangle - \langle \zeta a_n, a_n \zeta \rangle - \langle a_n \zeta, \zeta a_n \rangle + \langle \zeta a_n, \zeta a_n \rangle \\ &= 2\lambda(a_n^* a_n). \end{aligned}$$

Eq. (4.1) shows that $\|a_n \zeta - \zeta a_n\|_2^2 \not\rightarrow 0$ as $n \rightarrow \infty$, which is a contradiction.

The last statement follows from the above argument by considering compressions of \mathcal{M} by projections in A , because, for any nonzero projection $p \in A$, identifying p as the indicator of a measurable set E_p , it follows that the left-right-measure of the inclusion $Ap \subset p\mathcal{M}p$ will be the class of the restriction of $\lambda \otimes \lambda + \mu$ to $E_p \times E_p$. \square

The next result is a generalization of Prop. 4.2. We skip its proof, as the proof is similar to the proof of Prop. 4.2.

Proposition 4.3. *Let $A \subset \mathcal{M}$ be a masa. Let the left-right-measure of A restricted to the projection $pJqJ$ contain the product measure as a summand, where p and q are nonzero projections in A . Then:*

- (i) $\Gamma(A) < 1$.
- (ii) *If $r \geq p, q$ is any projection in A , then Ar cannot contain nontrivial centralizing sequences of $r\mathcal{M}r$.*

Proposition 4.4. *Let $A \subset \mathcal{M}$ be a masa. Let the left-right-measure of A be $[\nu + \mu]$, where $\mu \perp \lambda \otimes \lambda$, $\nu \ll \lambda \otimes \lambda$, ν and μ are finite and $\nu \neq 0$. Then A cannot contain any centralizing sequence of \mathcal{M} consisting of weakly null unitaries.*

Proof. Without loss of generality, we can assume that $f = \frac{d\nu}{d(\lambda \otimes \lambda)} \in L^2(\lambda \otimes \lambda)$. Write $[0, 1] \times [0, 1] \setminus \Delta([0, 1]) = E \cup F$, where $\nu(E) = 0$ and $\mu(F) = 0$. There exists a nonzero vector $\zeta_0 \in L^2(\mathcal{M}) \ominus L^2(A)$ such that for $a, b \in C[0, 1]$,

$$\eta_{\zeta_0}(a \otimes b) = \int_{[0, 1] \times [0, 1]} a(t)b(s)f(t, s)d\lambda(t)d\lambda(s).$$

The direct integral of ζ_0 is supported on F . Arguing as in the proof of Thm. 2.7, we conclude that $\mathbb{E}_A(\zeta_0 b \zeta_0^*) \in L^2(A)$ for all $b \in C[0, 1]$ and $\sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(\zeta_0 v^k \zeta_0^*)\|_2^2 < \infty$, where $v \in A$ is the Haar unitary generator corresponding to the function $t \mapsto e^{2\pi i t}$.

Suppose to the contrary, there is a sequence $\{a_n\} \subset C[0, 1] \subset A$ of weakly null unitaries that centralize \mathcal{M} . Given $\epsilon > 0$, choose $k_0 \in \mathbb{N}$ such that $\sum_{|k| \geq k_0} \|\mathbb{E}_A(\zeta_0 v^k \zeta_0^*)\|_2^2 < \epsilon^2$. Therefore on one hand,

$$\|\mathbb{E}_A(\zeta_0 a_n \zeta_0^*)\|_1 = \left\| \sum_{k \in \mathbb{Z}} \tau(a_n v^{-k}) \mathbb{E}_A(\zeta_0 v^k \zeta_0^*) \right\|_1$$

$$\begin{aligned}
&\leq \left\| \sum_{|k| < k_0} \tau(a_n v^{-k}) \mathbb{E}_A(\zeta_0 v^k \zeta_0^*) \right\|_1 + \left\| \sum_{|k| \geq k_0} \tau(a_n v^{-k}) \mathbb{E}_A(\zeta_0 v^k \zeta_0^*) \right\|_1 \\
&\leq \sum_{|k| < k_0} \left| \tau(a_n v^{-k}) \right| \left\| \mathbb{E}_A(\zeta_0 v^k \zeta_0^*) \right\|_1 + \sum_{|k| \geq k_0} \left| \tau(a_n v^{-k}) \right| \left\| \mathbb{E}_A(\zeta_0 v^k \zeta_0^*) \right\|_2 \\
&\leq \sum_{|k| < k_0} \left| \tau(a_n v^{-k}) \right| \left\| \mathbb{E}_A(\zeta_0 v^k \zeta_0^*) \right\|_1 + \left(\sum_{|k| \geq k_0} \left| \tau(a_n v^{-k}) \right|^2 \right)^{\frac{1}{2}} \left(\sum_{|k| \geq k_0} \left\| \mathbb{E}_A(\zeta_0 v^k \zeta_0^*) \right\|_2^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{|k| < k_0} \left| \tau(a_n v^{-k}) \right| \left\| \mathbb{E}_A(\zeta_0 v^k \zeta_0^*) \right\|_1 + \epsilon.
\end{aligned}$$

Since $a_n \xrightarrow{w.o.t} 0$ and ϵ is arbitrary, so $\|\mathbb{E}_A(\zeta_0 a_n \zeta_0^*)\|_1 \rightarrow 0$. On the other hand, $\|a_n^* \zeta_0 a_n \zeta_0^* - \zeta_0 \zeta_0^*\|_1 \rightarrow 0$ as $n \rightarrow \infty$ and consequently $\|a_n^* \mathbb{E}_A(\zeta_0 a_n \zeta_0^*) - \mathbb{E}_A(\zeta_0 \zeta_0^*)\|_1 \rightarrow 0$. So $\|\mathbb{E}_A(\zeta_0 a_n \zeta_0^*)\|_1 \rightarrow \|\mathbb{E}_A(\zeta_0 \zeta_0^*)\|_1$. This is a contradiction as ζ_0 is nonzero. \square

Corollary 4.5. *The left-right-measure of $A(1)$ is singular with respect to the product class.*

Proof. By construction $A(1)$ contains a centralizing sequence of weakly null unitaries. \square

Corollary 4.6. *Every strongly stable (McDuff) factor contains a singular masa whose left-right-measure is singular with respect to the product class.*

Proof. For existence of a singular masa in a II_1 factor see [20]. The statement follows by tensoring any singular masa in the factor by the alternating Tauer masa in \mathcal{R} (see [20, 28], Prop. 5.2 [8] and Lemma 3.5 [12]). \square

It is now natural to ask the following question. If $\mathcal{M} = M_1 \overline{\otimes} M_2$, where both M_1, M_2 are II_1 factors then does \mathcal{M} contain a (singular) masa whose *left-right-measure* is singular with respect to the product class?

Definition 4.7. A finite measure μ on $[0, 1]$ (or S^1) is called α -rigid for $|\alpha| = 1$, if and only if, there is a subsequence $\hat{\mu}_{n_k}$ of $\hat{\mu}_n = \int_0^1 e^{-2\pi i n t} d\mu(t)$ (or $\hat{\mu}_n = \int_{S^1} z^{-n} d\mu(z)$) that converges to $\alpha\mu([0, 1])$ (or $\alpha\mu(S^1)$) as $k \rightarrow \infty$. A 1-rigid measure is called rigid or a Dirichlet measure.

We now recall some properties of α -rigid measures. For details check Ch.7 [13]. Let μ be a α -rigid measure on $[0, 1]$. Any sequence n_k along which $\hat{\mu}_{n_k}$ converges to $\alpha\mu([0, 1])$ is said to be a sequence associated with μ . It is easy to see that, μ is α -rigid, if and only if, the sequence of functions $[0, 1] \ni t \mapsto e^{-2\pi i n_k t}$ converges to α in μ -measure. Thus ν is α -rigid with associated sequence n_k for any $\nu \ll \mu$. So α -rigidity is a property of equivalence class of measures, and hence can be thought of as a property of unitary operators, by considering appropriate Koopman operators. Atomic measures are always rigid.

To motivate what follows, we consider rigid m.p. transformations. Let T be a m.p. automorphism of a standard probability space (X, μ) . Let U_T denote the associated Koopman operator on $L^2(X, \mu)$. The transformation T is said to be rigid if $1 \in \overline{\{U_T^n\}_{n \in \mathbb{Z}} \setminus \{0\}}^{s.o.t}$ [14].

Assume further that T is weakly mixing. Then $L(\mathbb{Z}) \subset L^\infty(X, \mu) \rtimes_T \mathbb{Z}$ is a singular masa [15] (also see Thm. 2.1 [14]). Let $U_T^{n_k} \xrightarrow{s.o.t} 1$ as $k \rightarrow \infty$. A simple calculation shows that $L(\mathbb{Z})$ contains a centralizing sequence of the crossed product factor consisting of powers of the standard Haar unitary generator. It is not known whether this is always the case for Γ -masas. Let ν (which is a measure on $\widehat{\mathbb{Z}} = S^1$) denote the maximal spectral type of the action T . Then there is a unit vector $f \in L^2(X, \mu)$ such that $\hat{\nu}_n = \langle U_T^n f, f \rangle$ for all $n \in \mathbb{Z}$. It follows that ν is a Dirichlet measure. The relationship between the maximal spectral type of an action and the *left-right measure* of the associated masa appeared in Prop. 3.1 [14]. Thus by general theory of α -rigid measures (see Ch. 7 [13]), it follows that for λ almost all t (λ is Haar measure), the measure $\tilde{\eta}^t$ is α -rigid for all $\alpha \in S^1$.

In the general case, when A contains a nontrivial centralizing sequence of \mathcal{M} , one can choose a

central sequence consisting of trigonometric polynomials without constant term. We do not know whether we can choose a central sequence of the form $t \mapsto e^{2\pi i n_k t}$. In case we can, results analogous to the crossed product situation hold.

Making appropriate changes to the proof of Lemma 2.2, we get the following result. Its proof uses basic facts about L^1 spaces associated to finite von Neumann algebras. We omit its proof.

Lemma 4.8. *Let $\zeta \in L^2(\mathcal{M})$ be such that $\mathbb{E}_A(\zeta) = 0$. Let η_ζ denote the measure on $[0, 1] \times [0, 1]$ defined in Eq. (1.1). Let $b, w \in C[0, 1]$. Then*

$$\|\mathbb{E}_A(b\zeta w\zeta^*)\|_1 = \int_0^1 |b(t)| |\eta_\zeta^t(1 \otimes w)| d\lambda(t).$$

Theorem 4.9. *Let $A \subset \mathcal{M}$ be a singular masa. Let $v \in A$ be a Haar unitary generator of A . Suppose there exists a subsequence n_k ($n_k < n_{k+1}$ for all k) such that for all $y \in \mathcal{M}$,*

$$\|v^{n_k}y - yv^{n_k}\|_2 \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Then the measure $\tilde{\eta}^t$ is β -rigid for all $\beta \in S^1$, λ almost all t , where $[\eta]$ is the left-right-measure of A .

Proof. Let w be the Haar unitary generator of A that corresponds to the function $[0, 1] \ni t \mapsto e^{2\pi i t}$. The map from $L^\infty([0, 1], \lambda)$ to itself, which sends v^n to w^n for $n \in \mathbb{Z}$, implements a m.p. Borel isomorphism $T : [0, 1] \mapsto [0, 1]$. Then $T \times T$ implements an unitary $U : L^2(\mathcal{M}) \mapsto L^2(\mathcal{M})$, which preserves the structure of $L^2(\mathcal{M})$ as the natural A, A -bimodule (see Defn. 1.3). Standard density arguments show that if $\xi \in L^2(\mathcal{M})$, then

$$\|v^{n_k}\xi - \xi v^{n_k}\|_2 \rightarrow 0 \text{ as } k \rightarrow \infty.$$

So, we can assume $v = w$.

We know that there is a nonzero vector $\zeta \in L^2(\mathcal{M}) \ominus L^2(A)$ such that $\eta = \eta_\zeta$. Therefore $\|\mathbb{E}_A(v^{-n_k}\zeta v^{n_k}\zeta^*) - \mathbb{E}_A(\zeta\zeta^*)\|_1 \rightarrow 0$ as $k \rightarrow \infty$. Consequently, using similar arguments that are needed to prove Lemma 4.8, we have,

$$\|\mathbb{E}_A(v^{-n_k}\zeta v^{n_k}\zeta^*) - \mathbb{E}_A(\zeta\zeta^*)\|_1 = \int_0^1 |e^{-2\pi i n_k t} \eta^t(1 \otimes v^{n_k}) - \mathbb{E}_A(\zeta\zeta^*)(t)| d\lambda(t) \rightarrow 0$$

as $k \rightarrow \infty$. Hence, there exists a further subsequence n_{k_l} and a subset $E \subset [0, 1]$ such that $\lambda(E) = 0$, and for $t \in E^c$,

$$(4.2) \quad e^{-2\pi i n_{k_l} t} \eta^t(1 \otimes v^{n_{k_l}}) - \mathbb{E}_A(\zeta\zeta^*)(t) \rightarrow 0 \text{ as } l \rightarrow \infty.$$

Note that $\mathbb{E}_A(\zeta\zeta^*)(t) = \tilde{\eta}^t([0, 1]) < \infty$ (see Lemma 2.2) almost everywhere λ .

It is known that for almost every $\beta \in [0, 1]$ (with respect to λ), the set of limit points of the sequence $e^{-2\pi i n_{k_l} \beta}$ contains a point of the form $e^{2\pi i \alpha}$ with α irrational (see Ch. 7 [13]). Thus by enlarging the null set E and renaming it to be E again, we conclude that $e^{-2\pi i n_{k_l} t} \rightarrow e^{2\pi i \alpha t}$ for $t \in E^c$, α_t irrational. The subsequence in the last statement depends on t . By a diagonal argument, it follows that for $t \in E^c$, the measure $\tilde{\eta}^t$ is β -rigid for all $\beta \in S^1$ (see Ch. 7 [13]). \square

Remark 4.10. Examples of singular masas in the hyperfinite II_1 factor can be constructed that satisfy the hypothesis of Thm. 4.9. There exist weakly mixing actions of a stationary Gaussian process that has the desired properties (check §5 [31]). In fact, if A is Cartan in \mathcal{R} , then there is a centralizing sequence in A consisting of powers of some Haar unitary generator. This follows from Thm. 5.5 [12], Prop. 3.1 [14], Thm. 4 [31] and [4]. For example, consider the Cartan masa in the hyperfinite II_1 factor which arises from a irrational rotation along the direction of the group.

5. EXAMPLES OF SINGULAR MASAS IN THE FREE GROUP FACTORS

In this section, we show that given any subset S of \mathbb{N} , there are uncountably many pairwise non conjugate singular masas in $L(\mathbb{F}_k)$, $k \geq 2$, with Pukánszky invariant $S \cup \{\infty\}$. All examples exhibited in this section are constructed from examples appearing in [8, 26]. For any masa A considered in this section, we assume $A = L^\infty([0, 1], \lambda)$, where λ is the Lebesgue measure. If $A \subset \mathcal{M}$ is a masa and $[\eta]$ is its left-right-measure, then we will most of the time assume that $\eta(\Delta[0, 1]) = 0$. There are few exceptions to this assumption in this section, in which case we shall notify accordingly. The next two corollaries are direct applications of results in [8] (see Lemma 3.1, Thm. 3.2, Lemma 5.7 and Prop.

5.10 of [8]). The singularity of a masa $A \subset M \subset M * N$ (M, N are diffuse) as deduced in this section from results in [12], can also be deduced from Thm. 2.3 [8] or [19].

Corollary 5.1. *Let $k \in \mathbb{N}_\infty$ and $k \geq 2$. Let $A \subset L(\mathbb{F}_k)$ be a masa. If A is freely complemented then $\text{Puk}(A) = \{\infty\}$ and its left-right-measure is the class of product measure. In particular, A is singular.*

Proof. Follows directly from Lemma 5.7 and Prop. 5.10 [8]. Singularity follows from Cor. 3.2 in [21]. \square

Corollary 5.2. *Let $k \in \mathbb{N}_\infty$ and $k \geq 2$. Let $A \subset L(\mathbb{F}_k)$ be a masa. Let $A \subsetneq B \subsetneq L(\mathbb{F}_k)$, where B is a subalgebra and B is freely complemented.*

- (i) *If the left-right-measure $[\eta_B]$ of the inclusion $A \subset B$ is singular with respect to $\lambda \otimes \lambda$, then $\text{Puk}_{L(\mathbb{F}_k)}(A) = \text{Puk}_B(A) \cup \{\infty\}$ and the left-right-measure of the inclusion $A \subset L(\mathbb{F}_k)$ is $[\eta_B + \lambda \otimes \lambda]$.*
- (ii) *If the left-right-measure $[\eta_B]$ of the inclusion $A \subset B$ is $[\lambda \otimes \lambda]$, then $\text{Puk}_{L(\mathbb{F}_k)}(A) = \{\infty\}$ and the left-right-measure of the inclusion $A \subset L(\mathbb{F}_k)$ is $[\lambda \otimes \lambda]$.*

Proof. Follows from Lemma 5.7 and Prop 5.10 [8]. \square

Let T be a nonempty subset of \mathbb{N} . Let $T = \{n_k\}$ with $n_1 < n_2 < \dots$. Define

$$(5.1) \quad \mathbb{P}_T = \left\{ \alpha = \{\alpha_{n_k}\}_{k=1}^{|T|} : \alpha_{n_k} > \alpha_{n_{k+1}}, 0 < \alpha_{n_k} < 1 \text{ for all } k, \sum_{k=1}^{|T|} \alpha_{n_k} = 1 \right\}.$$

For $\alpha, \beta \in \mathbb{P}_T$, we say $\alpha \neq \beta$ if $\alpha_{n_k} \neq \beta_{n_k}$ for some k .

Theorem 5.3. *Let $B \subset \mathcal{R}$ be a singular masa such that the left-right-measure $[\eta_B]$ of the inclusion $B \subset \mathcal{R}$ is singular with respect to $\lambda \otimes \lambda$. For each $\alpha \in \mathbb{P}_T$, there exists a singular masa $B_\alpha \subset L(\mathbb{F}_2)$ with $\text{Puk}_{L(\mathbb{F}_2)}(B_\alpha) = \text{Puk}_{\mathcal{R}}(B) \cup \{\infty\}$. If $\alpha \neq \beta$ are any two elements of \mathbb{P}_T , then B_α and B_β are not conjugate.*

Proof. Fix $\alpha \in \mathbb{P}_T$. Let $\mathcal{R}_\alpha = \bigoplus_{n=1}^\infty \mathcal{R}$. Equip \mathcal{R}_α with the faithful trace

$$\tau_{\mathcal{R}_\alpha}(\cdot) = \sum_{n=1}^\infty \alpha_n \tau_{\mathcal{R}}(\cdot),$$

where $\tau_{\mathcal{R}}$ denotes the unique normal tracial state of \mathcal{R} .

Then $B_\alpha = \bigoplus_{n=1}^\infty B$ is a singular masa in the hyperfinite algebra \mathcal{R}_α and the latter is separable. The projections $(0 \oplus \dots \oplus 0 \oplus 1 \oplus 0 \oplus \dots)$, where 1 appears at the n -th coordinate, is a central projection p_n of \mathcal{R}_α and it belongs to B_α . The projections p_n correspond to the indicator function of measurable subsets $F_n \subset ([0, 1], \lambda)$ respectively, so that $F_n \cap F_m$ is a set of λ measure 0 for all $n \neq m$. By applying appropriate transformations, the left-right-measure of $B \subset \mathcal{R}$ can be transported to each $F_n \times F_n$, which we denote by $[\eta_n]$. We also assume $\eta_n(F_n \times F_n) = 1$ for all n . Note that by factoriality of \mathcal{R} it follows that $\eta_n(E \times F) > 0$ for all measurable rectangles $E \times F \subset F_n \times F_n$ such that $\lambda(E) > 0$, $\lambda(F) > 0$ (Lemma 2.9 [12]).

Consider $(\mathcal{M}, \tau_{\mathcal{M}}) = (\mathcal{R}_\alpha, \tau_{\mathcal{R}_\alpha}) * (\mathcal{R}, \tau_{\mathcal{R}})$. Then \mathcal{M} is isomorphic to $L(\mathbb{F}_2)$ by a well known theorem of Dykema [7]. Then $B_\alpha \subset L(\mathbb{F}_2)$ is a singular masa by Thm. 2.3 [8]. The left-right-measure of the inclusion $B_\alpha \subset L(\mathbb{F}_2)$ is

$$[\lambda \otimes \lambda + \sum_{n=1}^\infty \frac{1}{2^n} \eta_n]$$

and $\text{Puk}_{L(\mathbb{F}_2)}(B_\alpha) = \bigcup_n \text{Puk}_{(\mathcal{R} * \mathcal{R})}(B) \cup \{\infty\} = \text{Puk}_{\mathcal{R}}(B) \cup \{\infty\}$ from Cor. 5.2 and Thm. 3.2 [8].

Since automorphisms of II_1 factors preserve the trace and orthogonal projections, the non conjugacy of B_α and B_β for $\alpha \neq \beta$ follows by considering the left-right-measures. Indeed, if $B_\beta = \phi(B_\alpha)$ for some automorphism ϕ of $L(\mathbb{F}_2)$ and $\alpha, \beta \in \mathbb{P}_T$, then B_α and B_β would have identical bimodule structure. Therefore, there is a Borel isomorphism $\tilde{\phi} : [0, 1] \mapsto [0, 1]$ such that $\tilde{\phi}_* \lambda = \lambda$ and $(\tilde{\phi} \times \tilde{\phi})_* [\eta_{B_\alpha}] = [\eta_{B_\beta}]$, where $\eta_{B_\alpha}, \eta_{B_\beta}$ denote the left-right-measures of B_α and B_β respectively.

Let $F_n, E_n, n = 1, 2, \dots$, be the measurable partitions of $[0, 1]$ associated to the left-right-measures of B_α and B_β (as described above) respectively. Clearly, the class of the singular part of η_{B_α} will be

pushed forward to the class of the singular part of η_{B_β} . If we let $\chi_{E'_n} = \phi(\chi_{F_n})$, then $E'_n = E_{k_n} \bmod \lambda$ for some k_n . But the λ -measure of F_n and hence E_{k_n} are strictly decreasing as n increases. Thus $k_n = n$ for all n . This completes the proof. \square

Now we construct non conjugate singular masas in the free group factors which have the same multiplicity. We will give a case by case argument.

Case: $\{1, \infty\}$: In Thm. 5.3, let B be the alternating Tauer masa $A(1)$.

Case: $\{1, n, \infty\}$, $n \neq 1$: Consider the matrix groups

$$(5.2) \quad G_n = \left\{ \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} \mid f \in P_n, x \in \mathbb{Q} \right\}, \quad H_n = \left\{ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \mid f \in P_n \right\} \subset G_n, \text{ where}$$

$$P_\infty = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, p, q \text{ odd} \right\} \text{ and } P_n = \left\{ f 2^{kn} \mid f \in P_\infty, k \in \mathbb{Z} \right\} \text{ if } n < \infty,$$

are subgroups of the multiplicative group of nonzero rational numbers. Then $L(G_n)$ is the hyperfinite II_1 factor \mathcal{R} and $L(H_n) \subset L(G_n)$ is a singular masa with *Pukánszky invariant* $\{n\}$. The *left-right-measure* of the inclusion $L(H_n) \subset L(G_n)$ is the class of product Haar measure $\lambda_{\widehat{H}_n} \otimes \lambda_{\widehat{H}_n}$ on $\widehat{H}_n \times \widehat{H}_n$, where \widehat{H}_n denotes the character group of H_n (see Example 5.1 [26] and Example 6.2 [8]).

As $\mathcal{R} \overline{\otimes} \mathcal{R} \cong \mathcal{R}$, so $L(H_n) \overline{\otimes} A(1)$ is a singular masa in \mathcal{R} from [28] (see also [3] and Thm. 5.15 [12]). Let $A(1) = L^\infty([0, 1], \lambda)$. The *Pukánszky invariant* of the inclusion $L(H_n) \overline{\otimes} A(1) \subset \mathcal{R}$ is $\{1, n\}$ from Thm. 2.1 [26]. The *left-right-measure* of the inclusion $L(H_n) \overline{\otimes} A(1) \subset \mathcal{R}$ is the class of

$$\lambda_{\widehat{H}_n} \otimes \lambda_{\widehat{H}_n} \otimes \eta + \tilde{\Delta}_* \lambda_{\widehat{H}_n} \otimes \eta + \lambda_{\widehat{H}_n} \otimes \lambda_{\widehat{H}_n} \otimes \tilde{\Delta}_* \lambda,$$

on $\widehat{H}_n \times \widehat{H}_n \times [0, 1] \times [0, 1]$, where $[\eta]$ is the *left-right-measure* of the *alternating Tauer masa* restricted to the off diagonal and $\tilde{\Delta}$ is the map, that maps a set to its square by sending $x \mapsto (x, x)$ (see Prop. 5.2 [8]). *In this case, we need to specify the measures on the diagonals as they are necessary.* Given $\alpha \in \mathbb{P}_\mathbb{N}$, replace the role of B in Thm. 5.3 by $L(H_n) \overline{\otimes} A(1)$ to construct a masa $A_{\alpha, n} \subset L(\mathbb{F}_2)$.

Case: $\{n, \infty\}$, $n \neq 1$: Let $H_n \subset G_n$ and $H_\infty \subset G_\infty$ be as in the previous case. Then $L(H_n \times H_\infty)$ is a singular masa in $L(G_n \times G_\infty)$ whose *measure-multiplicity invariant* is the equivalence class of

$$(\widehat{H}_n \times \widehat{H}_\infty, \lambda_{\widehat{H}_n} \otimes \lambda_{\widehat{H}_\infty}, [\eta], m),$$

where η is the sum of

- (i) Haar measure on $(\widehat{H}_n \times \widehat{H}_\infty) \times (\widehat{H}_n \times \widehat{H}_\infty)$;
- (ii) Haar measure on the subgroup

$$D_\infty = \{(\alpha, \beta_1, \alpha, \beta_2) \mid \alpha \in \widehat{H}_n, \beta_1, \beta_2 \in \widehat{H}_\infty\};$$

- (iii) Haar measure on the subgroup

$$D_n = \{(\alpha_1, \beta, \alpha_2, \beta) \mid \alpha_1, \alpha_2 \in \widehat{H}_n, \beta \in \widehat{H}_\infty\};$$

and where the multiplicity function on the off-diagonal is given by

$$m(\gamma) = \begin{cases} n, & \gamma \in D_n \setminus \Delta(\widehat{H}_n \times \widehat{H}_\infty), \\ \infty, & \text{otherwise.} \end{cases}$$

This was calculated in §6, [8]. Note that η contain singular summands, singular with respect to product Haar measure off the diagonal $\Delta(\widehat{H}_n \times \widehat{H}_\infty)$. For each $\alpha \in \mathbb{P}_\mathbb{N}$, make a construction analogous to Thm. 5.3 with B replaced by $L(H_n) \overline{\otimes} L(H_\infty)$, to construct a masa $A_{\alpha, n} \subset L(\mathbb{F}_2)$. Note that $\text{Puk}_{L(\mathbb{F}_2)}(A_{\alpha, n}) = \{n, \infty\}$ from Thm. 3.2 and Lemma 5.7 [8]. The *left-right-measure* of the inclusion $A_{\alpha, n} \subset L(\mathbb{F}_2)$ is of the same form as discussed in the previous cases. Use Lemma 3.6, Thm. 3.8 of [12] to decide non conjugacy of $A_{\alpha, n}, A_{\beta, n}$ whenever $\alpha \neq \beta \in \mathbb{P}_\mathbb{N}$.

Case: $S \cup \{\infty\}$, $S \subseteq \mathbb{N}$, $1 \in S$ and $|S| \geq 2$: Write $S = \{n_k : 1 = n_1 < n_2 < \dots\}$. Let P_n and P_∞ be the subgroups of the multiplicative group of rational numbers as before. Let G_n , $n \geq 1$, be the matrix group

$$G_n = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix} : x, y \in \mathbb{Q}, f \in P_n, g \in P_\infty \right\}$$

and H_n the subgroup consisting of the diagonal matrices in G_n . Then as noted in Example 5.2 of [26], G_n is amenable and $L(G_n) \cong \mathcal{R}$. It is also true that $L(H_n)$ is a singular masa in $L(G_n)$ with

Pukánszky invariant $\{n, \infty\}$ (see Prop. 2.5 [8]). Consider $M_n = L(G_n) \overline{\otimes} \mathcal{R} \cong \mathcal{R}$ and consider the masa $A_n = L(H_n) \overline{\otimes} A(1)$. Then $A_n \subset M_n$ is a singular masa with $\text{Puk}_{M_n}(A_n) = \{1, n, \infty\}$ (see Thm. 2.1 [26]). Fix $\alpha \in \mathbb{P}_S$. Now consider

$$\mathcal{M}_\alpha = \oplus_{n \in S} M_n \text{ and } A_\alpha = \oplus_{n \in S} A_n, \text{ where } \tau_{\mathcal{M}_\alpha}(\cdot) = \sum_{n \in S} \alpha_n \tau_{M_n}(\cdot),$$

where τ_{M_n} denotes the faithful normal tracial state of M_n . Then $\mathcal{M}_\alpha * L(\mathbb{Z}) = \mathcal{M}_\alpha * L(\mathbb{F}_1) \cong L(\mathbb{F}_2)$ [7], A_α is a singular masa in $L(\mathbb{F}_2)$ and $\text{Puk}_{L(\mathbb{F}_2)}(A_\alpha) = S \cup \{\infty\}$ from Thm. 3.2 [8].

There exist orthogonal projections $\{p_n\}_{n \in S} \subset A_\alpha$ with the property that $\sum_{n \in S} p_n = 1$ and $\tau_{L(\mathbb{F}_2)}(p_n) = \alpha_n$, such that the *left-right-measure* of the inclusion $A_\alpha \subset L(\mathbb{F}_2)$ has $\lambda \otimes \lambda$ as a summand and measures singular with respect to $\lambda \otimes \lambda$ on the squares $p_n \times p_n$ (here by abuse of notation we think of p_n as a measurable set which corresponds to the projection p_n). The singular part on $p_n \times p_n$ has the property that its $(\pi_1, \lambda), (\pi_2, \lambda)$ disintegrations are non zero almost everywhere on p_n . Non conjugacy of A_α and A_β for $\alpha \neq \beta$ follows easily from Lemma 3.6, Thm. 3.8 of [12].

Case: $S \cup \{\infty\}$, $S \subseteq \mathbb{N}$, $1 \notin S$ and $|S| \geq 2$: Let G_n, H_n for $n \in \mathbb{N}_\infty$ be the groups defined in Eq. (5.2). Let $M_n = L(G_n \times G_\infty)$ and $A_n = L(H_n \times H_\infty)$ for $n \in S$. Fix $\alpha \in \mathbb{P}_S$. Let $\mathcal{M}_{\alpha, S} = \oplus_{n \in S} M_n$ and $A_{\alpha, S} = \oplus_{n \in S} A_n$, where $\mathcal{M}_{\alpha, S}$ is equipped with the trace $\tau_{\mathcal{M}_{\alpha, S}}(\cdot) = \sum_{n \in S} \alpha_n \tau_{M_n}(\cdot)$, where τ_{M_n} denotes the faithful normal tracial state of M_n . Replace the role of the masa A_α in the previous case by $A_{\alpha, S}$. We omit the details.

Case: $\{\infty\}$: Consider the hyperfinite II_1 factor \mathcal{R} with a singular masa A such that $\text{Puk}_{\mathcal{R}}(A) = \{\infty\}$. Consider the inclusion $B = \overline{\otimes}_{n=1}^\infty A \subset \overline{\otimes}_{n=1}^\infty \mathcal{R}$. Since up to isomorphism, there is one hyperfinite II_1 factor, so $B \subset \mathcal{R}$ is a masa from Tomita's theorem on commutants. Since $\text{Puk}(B) = \{\infty\}$ from Lemma 2.4 [33], so B is singular from Cor. 3.2 [21]. Clearly, $\Gamma(B) = 1$. The *left-right-measure* of the inclusion $B \subset \mathcal{R}$ is singular to the product class from Thm. 4.4. Now apply Thm. 5.3.

The above constructions lead to the following theorem.

Theorem 5.4. *Let S be an arbitrary (could be empty) subset of \mathbb{N} . Let $k \in \{2, 3, \dots, \infty\}$ and let Γ be any countable discrete group. There exist uncountably many pairwise non conjugate singular masas in $L(\mathbb{F}_k * \Gamma)$ whose Pukánszky invariant is $S \cup \{\infty\}$.*

Proof. We have already proved that there exist uncountably many pairwise non conjugate singular masas $\{A_\alpha\}_{\alpha \in I}$, where I is some indexing set, in $L(\mathbb{F}_2)$ whose Pukánszky invariant is $S \cup \{\infty\}$. One has isomorphisms [7]

$$L(\mathbb{F}_2) * L(\mathbb{F}_{k-2} * \Gamma) \cong L(\mathbb{F}_k * \Gamma) \text{ for } k \geq 2.$$

For $k \geq 2$, each A_α is a singular masa in $L(\mathbb{F}_k * \Gamma)$ [8]. Use Lemma 5.7, Prop. 5.10 [8] to decide the non conjugacy of A_α and A_β when $\alpha \neq \beta$, in the free product. \square

Theorem 5.5. *There exist non conjugate singular masas A, B in $L(\mathbb{F}_k)$, $2 \leq k \leq \infty$, with same measure-multiplicity invariant.*

Proof. Let $\mathcal{R} = L^\infty(\prod_{n \in \mathbb{Z}} (\{0, 1\}, \mu)) \rtimes \mathbb{Z}$, where $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$ and the action is Bernoulli shift. Then the copy of \mathbb{Z} gives rise to a masa $A \subset \mathcal{R}$ whose multiplicity is $\{\infty\}$ and whose *left-right-measure* is the class of product measure. This follows from the fact that the maximal spectral type of Bernoulli action is Lebesgue measure and its multiplicity is infinite on the orthocomplement of constant functions and Prop. 3.1 [14]. Consequently, for $k \geq 2$, $A \subset \mathcal{R} * \left(\begin{smallmatrix} * \\ * \end{smallmatrix} \mathcal{R} \right)_{r=1}^{k-1} \cong L(\mathbb{F}_k)$ [7] is a singular masa whose *left-right-measure* is the class of product measure and whose multiplicity function is $m \equiv \infty$, off the diagonal. Let B be any single generator masa of $L(\mathbb{F}_k)$. The same holds for the single generator masas as well due to malnormality of group inclusions. A is not conjugate to B , as the former is not maximally injective, while the single generator masas are maximally injective from Cor. 3.3 [18]. \square

Remark 5.6. We end this paper with the following observation. The following example was constructed by Smith and Sinclair in Example 5.1 of [26]. It produces an example of a m.p. dynamical system which solves Banach's problem with the group under consideration being \mathbb{Q}^\times . Consider the

matrix groups $G = \left\{ \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} : f \in \mathbb{Q}^\times, x \in \mathbb{Q} \right\}$ and H the subgroup of G consisting of diagonal matrices. Then $L(G)$ is isomorphic to the hyperfinite II_1 factor, and, $L(H)$ is a singular masa in $L(G)$. Note that H is a malnormal subgroup of G and $G = N \rtimes H$, where $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Q} \right\}$. Matrix multiplication shows that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a bicyclic vector of $L(H)$. The *left-right-measure* of $L(H) \subset L(G)$ is the class of product measure (see Example 6.2 [8]).

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